

# Anomalies in Quantum Field Theory and Cohomologies of Configuration Spaces

Nikolay M. Nikolov

March 29, 2009

Institute for Nuclear Research and Nuclear Energy,  
Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria  
mitov@innr.bas.bg

## Abstract

In this paper we study systematically the Euclidean renormalization in configuration spaces. We investigate also the deviation from commutativity of the renormalization and the action of all linear partial differential operators. This deviation is the source of the anomalies in quantum field theory, including the renormalization group action. It also determines a Hochschild 1-cocycle and the renormalization ambiguity corresponds to a *nonlinear* subset in the cohomology class of this renormalization cocycle. We show that the related cohomology spaces can be reduced to de Rham cohomologies of the so called “(ordered) configuration spaces”. We find cohomological differential equations that determine the renormalization cocycles up to the renormalization freedom. This analysis is a first step towards a new approach for computing renormalization group actions. It can be also naturally extended to manifolds as well as to the case of causal perturbation theory.

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## 1 Introduction

The methods of the abstract algebra have been successfully applied for several decades to two dimensional conformal Quantum Field Theory (QFT). As a result, many nontrivial models have been found with a very rich and explicit structure. A question naturally arises to what extent it is possible to apply these algebraic methods to more general QFT and, for instance, to perturbative QFT. Let us motivate in more details this expectation. The two dimensional conformal field theories can be described by a purely algebraic structure called vertex algebra (see e.g., [11]). This structure has an analog in higher space-time dimensions and there it characterizes in a purely algebraic way the class of so called globally conformal invariant models of quantum fields ([13], [15]). Since the perturbative QFT can be considered as a deformation theory of vertex algebras then one may expect that the perturbation theory can be purely algebraically developed. On the other hand, more than half a century experience of perturbative QFT suggests that the usage of some transcendental methods are necessary even if we are perturbing free massless fields. The present work, in particular, arose from an attempt to understand the exact place and kind of the transcendental methods that are needed to derive the renormalization group action. We also consider the question what transcendental numbers appear in the perturbative expansion of the beta function. We conjecture, based on further investigations, which will be published elsewhere, that these transcendental numbers are multiple zeta values in any perturbative QFT on even space-time dimensions. Let us point also out that recently there are very intensive investigations of integrability in supersymmetric gauge theories. These expectations are related to the anomalous dimensions and hence, having an

algebraic approach to renormalization group will be very beneficial for these studies.

## 1.1 Motivation from perturbative QFT

Perturbation theory in QFT is one of the technically most difficult subjects in the contemporary theoretical physics. This is, first and foremost, due to the appearance of complicated integrals in higher orders, as well as, to the complexity of the accompanying renormalization. For the realistic QFT models there are practically no numerical results for arbitrary orders in perturbation theory. There are also a few methods that allow to perform calculations to all orders. Without pretending to give justice to various approaches to the subject we just point out the general analysis of perturbative renormalization in recent work of Connes-Kreimer (see e.g., [12, 5, 6]).

The present work is a first step to a new approach for determining the action of the renormalization group in perturbative QFT (i.e., for calculating beta functions). More generally, our analysis is also applicable to any anomalies in QFT. It offers in addition a geometric insight to the problem. The general idea of the method is to perform a cohomological analysis of the renormalization ambiguity and to use it to determine the renormalization group action. Furthermore, we separate the problem from the particular models of perturbative QFT, i.e. we consider all possible theories and even more general situations. It is this generality that makes the geometric interpretation possible. It is also important that we do not confine our treatment to the one parameter action of the renormalization group but consider all linear partial differential operators. This is done in order to restrict as much as possible the related cohomology owing to the general properties of the algebra of all differential operators. Our geometric view favors the study of renormalization in “coordinate space”. This approach has been originally developed by Bogolubov, and Epstein and Glaser [7] on Minkowski space and recently applied to more general pseudo-Riemann manifolds (see e.g., [4, 10]). It is also called causal perturbation theory. This approach has a counterpart in *Euclidean* QFT ([18], [9]), which is in some respects simpler. We choose to work here within this Euclidean framework, and even on  $\mathbb{R}^D$ , but our analysis can be extended to manifolds, as well as to the case of the causal perturbation theory on pseudo-Riemann manifolds. Our choice was motivated by the fact that the geometric structures appearing in the analysis are much more transparent in the Euclidean approach.

For the purpose of the present work we also systematically develop the theory of Euclidean renormalization on configuration spaces (Sect. 2). This

is done in the spirit of the Epstein–Glaser approach in the causal perturbative QFT but in contrast in the Euclidean case we deal only with the Green functions and there are no time-ordered (or retarded) products of fields. This makes different the axiomatic assumptions that are needed in order to achieve the “universal renormalization theorem”. The latter statement is the fact that the change of renormalization uniquely induces a change of the coupling constants as formal power series. (This is true for any theory, but only for the so called renormalizable theories a finite number of coupling constants remains under an arbitrary change of renormalization.) Let us point out that to the best of our knowledge there is no systematic treatment in the literature of Euclidean renormalization on configuration spaces. We split this theory in two parts: the first of them is model independent and we introduce there a general concept of renormalization maps as acting on algebras whose elements can be *bare* Feynman amplitudes of any theory. The remaining part of this study will treat the application of the renormalization maps to particular models of perturbative QFT. We intend to consider it in a future work.

We shall briefly explain the place of our analysis within the perturbative QFT.

In perturbative Euclidean QFT one computes the Green functions as formal power series of the type:

$$\begin{aligned}
 & G_N(z_1, \dots, z_N; \mathbf{g}; \{R_n\}) \\
 &= \sum_{\mathbf{r} \geq 0} \int_{\mathbb{R}^{D|\mathbf{r}|}} d^D x_1 \cdots d^D x_{|\mathbf{r}|} \frac{\mathbf{g}^{\mathbf{r}}}{\mathbf{r}!} R_{|\mathbf{r}|} \mathcal{A}_{\mathbf{r}}(x_1, \dots, x_{|\mathbf{r}|}; z_1, \dots, z_N). \quad (1.1)
 \end{aligned}$$

Here:  $x_k = (x_k^1, \dots, x_k^D)$  and  $z_k \in \mathbb{R}^D$ ;  $\mathbf{r} = (r_1, \dots, r_s)$ ,  $|\mathbf{r}| = \sum_j r_j$  and  $\mathbf{r}! = \prod_j r_j!$  are multiindex notations;  $\mathbf{g} = (g_1, \dots, g_s)$  is a system of coupling constants;<sup>1</sup>  $\mathcal{A}_{\mathbf{r}}(x_1, \dots, x_{|\mathbf{r}|}; z_1, \dots, z_N)$  are Feynman amplitudes (sums over Feynman graphs) with  $|\mathbf{r}|$  internal and  $N$  external vertices;  $\{R_n\}$  is a system of renormalization maps to be defined later. The important point for us in Eq. (1.1) is that after smearing the external points  $z_k$  of the Feynman amplitudes  $\mathcal{A}_{\mathbf{r}}(x_1, \dots, x_{|\mathbf{r}|}; z_1, \dots, z_N)$  by test functions they become finite sums of products of type:

$$\left( \prod_{1 \leq j < k \leq n} G_{jk}(x_j - x_k) \right) \left( \prod_{m=1}^n F_m(x_m) \right), \quad (1.2)$$

---

<sup>1</sup>or functions, with included space cutoff

where  $G_{jk}(\mathbf{x}_j - \mathbf{x}_k)$  are “propagators” and  $F_m(\mathbf{x}_m)$  are smooth functions on  $\mathbb{R}^D$ , which arise from the smearing of the external propagators ( $F(\mathbf{x}) = \int G(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d^D \mathbf{y}$ ). Since the propagators  $G_{jk}(\mathbf{x}_j - \mathbf{x}_k)$  are regular functions for  $\mathbf{x}_j \neq \mathbf{x}_k$ , the integrands of type (1.2) are well defined, regular functions on the subspace of all pairwise distinct arguments  $(\mathbf{x}_1, \dots, \mathbf{x}_n) \in (\mathbb{R}^D)^{\times n} (\cong \mathbb{R}^{Dn})$ . The latter subspace of  $\mathbb{R}^{Dn}$  is also called an *ordered configuration space* over  $\mathbb{R}^D$  and is denoted by  $F_n(\mathbb{R}^D)$ . The configuration spaces are generally introduced for arbitrary manifold (or set)  $X$  by:

$$F_n(X) = \left\{ (\mathbf{x}_1, \dots, \mathbf{x}_n) \in X^{\times n} : \mathbf{x}_j \neq \mathbf{x}_k \text{ if } j \neq k \right\}$$

and they are well studied (see e.g., [8]). Hence, the renormalization maps  $R_n$  in (1.1) are introduced in order to make the integrals well defined. In particular,  $R_n$  should extend smooth functions on configuration spaces to distributions over the whole space.<sup>2</sup> The system  $\{R_n\}$  have to satisfy certain natural properties that we shall consider in Sect. 2.

A perturbative QFT is said to be renormalizable if after changing the system of renormalization maps,  $\{R_n\} \rightarrow \{R'_n\}$ , there exists a unique formal power series

$$\alpha(\mathbf{g}) = \sum_{\mathbf{r} \geq 0} a_{\mathbf{r}} \frac{\mathbf{g}^{\mathbf{r}}}{\mathbf{r}!}$$

such that

$$G_N(z_1, \dots, z_N; \mathbf{g}; \{R'_n\}) = G_N(z_1, \dots, z_N; \alpha(\mathbf{g}); \{R_n\})$$

in the sense of formal power series. (More generally, one can consider all possible interactions each switched with its own coupling constant. Then the change of renormalization uniquely induces an action on the infinite set of coupling constants in the above sense of formal power series. This is the statement we called above “universal renormalization theorem”.)

In particular, one can change the renormalization maps by a dilation (i.e., changing the “scale”):

$$R_n^\lambda := d_\lambda \circ R_n \circ d_\lambda^{-1}$$

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<sup>2</sup>Let us point that we do not consider here the infrared problem, i.e., an additional extension related to the integration over an infinite volume (but it can be treated by the same method). This is because we shall be mainly interested in the renormalization group action (see below), which in QFT can be extracted only by its ultraviolet renormalization (or, its short distance properties).

(where  $(d_\lambda f)(x_1, x_2, \dots) := f(\lambda^{-1}x_1, \lambda^{-1}x_2, \dots)$  for  $\lambda \in \mathbb{R}^+$ ). As a result, this generates the action of renormalization group ( $R^+ \ni \lambda$ ) on the coupling constants

$$\{R_n^\lambda\} \rightarrow \alpha_\lambda(\mathbf{g}),$$

whose generator is the so called *beta function*

$$\beta(\mathbf{g}) = \lambda \frac{d}{d\lambda} \alpha_\lambda(\mathbf{g}) \Big|_{\lambda=1}.$$

Clearly, to compute the formal power series of  $\beta(\mathbf{g})$  one has to know the “commutators”

$$(\mathbf{x} \cdot \partial_{\mathbf{x}}) \circ R_n - R_n \circ (\mathbf{x} \cdot \partial_{\mathbf{x}}),$$

where  $\mathbf{x} \cdot \partial_{\mathbf{x}} := \sum_{k,\mu} x_k^\mu \frac{\partial}{\partial x_k^\mu}$ . We shall generalize this task and will look for the commutators

$$c_n[A] = A \circ R_n - R_n \circ A$$

for all linear partial differential operators  $A$ . It turns out that  $c_n[A]$  is a certain Hochschild cocycle and changing the renormalization corresponds to adding a coboundary (see Sect. 3). So, we would like to find some cohomological equations that would determine  $c_n[A]$ .

From our analysis in Sect. 2, it follows that the renormalization ambiguity allows us to achieve  $c_n[f] = 0$  for smooth functions  $f$  (i.e., differential operators of zeroth order). This allows us to extend our methods also for manifolds without even any metric structure on them. Moreover, the remaining nontrivial part of the cocycle  $c_n[\partial_{x_k^\mu}]$  ( $\partial_{x_k^\mu} := \frac{\partial}{\partial x_k^\mu}$ ) can be characterized by certain cohomological equations (Eqs. (3.6) and (3.35)). We prove in Theorem 3.1 that the cohomological ambiguity in the solutions of these equations exactly corresponds to the renormalization freedom. Then we reduce the related cohomologies to de Rham cohomologies of configuration spaces.

So, in the subsequent section we shall introduce the precise notion of renormalization maps. Then, we analyze the remaining nontrivial cohomological properties of the renormalization maps and their reduction to de Rham cohomologies. The essential material of Sect. 2 that is needed for Sect. 3 is contained in Sects. 2.1–2.4.

*Some common notations.*  $\mathbb{N} = \{1, 2, \dots\}$ ,  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ ,  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ ;  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are the fields of rational, real and complex numbers, respectively. For a finite set  $S$ ,  $|S|$  stands for the number of its elements. The vectors in Euclidean “space–time”  $\mathbb{R}^D$  ( $D$  standing for the space–time

dimension) are denoted by  $\mathbf{x} = (x^1, \dots, x^D), \mathbf{y}, \dots$ . Some times, we shall also deal with  $\mathbb{R}^N$  for other  $N \in \mathbb{N}$  (for instance,  $(\mathbb{R}^D)^{\times n} \equiv \mathbb{R}^{Dn}$ ) and then we shall denote its elements by  $\mathbf{x} = (x^1, \dots, x^N), \mathbf{y}, \dots$ . *Multiindex notations.* For  $\mathbf{r} = (r_1, \dots, r_N) \in \mathbb{N}_0^N$ :  $|\mathbf{r}| = \sum_j r_j$ ,  $\mathbf{r}! = \prod_j r_j!$ ,  $\mathbf{x}^{\mathbf{r}} = \prod_j (x^j)^{r_j}$  and  $\partial_{\mathbf{x}}^{\mathbf{r}} = \prod_j \partial_{x^j}^{r_j} (= \prod_j \left(\frac{\partial}{\partial x^j}\right)^{r_j})$ .

## 2 Theory of renormalization maps

### 2.1 Preliminary notions

In this subsection we shall introduce for every  $n = 2, 3, \dots$  an algebra  $\mathcal{O}_n$ , which should be thought of as the algebra of “ $n$ -point Feynman diagrams”. All these algebras are built by the algebra  $\mathcal{O} \equiv \mathcal{O}_2$ , which is “the algebra of propagators”. So, we shall fix only the propagator structure of the theory but not the vertex structure. In fact, even the propagators will be fixed only as a type of functions but not as an explicit system of functions (say, scalar propagators, spinor propagators etc.). For instance, if we perturb massless fields then the algebra  $\mathcal{O}$  is just the algebra of rational functions  $G(\mathbf{x})$  on the Euclidean space-time  $\mathbb{R}^D \ni \mathbf{x}$ , which denominators are only powers of the interval  $\mathbf{x}^2 := (x^1)^2 + \dots + (x^D)^2$ . (This algebra is denoted by  $\mathbb{Q}[\mathbf{x}, 1/\mathbf{x}^2]$  if the coefficients are rational numbers.) We shall introduce later the renormalization maps as certain linear maps acting on the algebras  $\mathcal{O}_n$ .

Since we wish to take into account what transcendental methods and numbers are needed to describe renormalization we shall fix some ground field  $\kappa \subseteq \mathbb{R}$ , which will be assumed to be the field  $\mathbb{Q}$  of rational numbers if not stated otherwise. The vector spaces and associative algebras will be usually assumed over  $\kappa$  (except for some standard spaces as the distributions spaces which will be considered on  $\mathbb{R}$ ).

All the algebras we shall use will be **differential algebras**. A differential algebra with  $N$ -derivatives is a *commutative* associative algebra endowed with linear operators  $t_j$  and  $\partial_{t_j}$  for  $j = 1, \dots, N$  such that, they satisfy the Heisenberg commutation relations:

$$t_j t_k - t_k t_j = \partial_{t_j} \partial_{t_k} - \partial_{t_k} \partial_{t_j} = 0, \quad t_j \partial_{t_k} - \partial_{t_k} t_j = \delta_{jk},$$

$\partial_{t_j}$  satisfy the Leibnitz rule, and  $t_j$  commute with the multiplication by the elements of the algebra:

$$\partial_{t_j}(ab) = \partial_{t_j}(a)b + a\partial_{t_j}b, \quad t_j(ab) = at_jb.$$

An example of differential algebra is the  $\mathbb{R}$ -algebra  $\mathcal{C}^\infty(\mathbb{R}^{Dn})$  of smooth functions  $f(x_1, \dots, x_n)$  over  $\mathbb{R}^{Dn}$ , where we shall denote the operators  $t_j$  and  $\partial_{t_j}$  by  $x_k^\mu$  and  $\partial_{x_k^\mu} \equiv \frac{\partial}{\partial x_k^\mu}$  ( $k = 1, \dots, n, \mu = 1, \dots, D$ ), respectively, but some times we shall also replace the pair of indices  $(k, \mu)$  with a single letter  $\xi$  and write  $x^\xi$  and  $\partial_{x^\xi}$ .

We shall focus our analysis on translation invariant  $n$ -point functions on the Euclidean space  $\mathbb{R}^D$  and for short we denote the quotient space:

$$\mathbf{E}_n := (\mathbb{R}^D)^{\times n} / \mathbb{R}^D \equiv \mathbb{R}^{Dn} / \mathbb{R}^D, \quad (2.1)$$

where the quotient is taken under the action of  $\mathbb{R}^D$  by translations:  $(x_1, \dots, x_n) \mapsto (x_1 + u, \dots, x_n + u)$ . Thus, we have an isomorphism

$$\mathbf{E}_n \cong \mathbb{R}^{D(n-1)} : [x_1, \dots, x_n] \mapsto (x_1 - x_n, \dots, x_{n-1} - x_n), \quad (2.2)$$

where  $[x_1, \dots, x_n]$  stands for a class  $(x_1, \dots, x_n) \bmod \mathbb{R}^D$ . Recall that  $F_n(\mathbb{R}^D)$  stands for the configuration space over  $\mathbb{R}^D$  and denote

$$\mathbf{F}_n := F_n(\mathbb{R}^D) / \mathbb{R}^D$$

where quotient is taken again under the above action of  $\mathbb{R}^D$  on  $\mathbb{R}^{Dn}$ . We have again an isomorphism

$$\mathbf{F}_n \cong F_{n-1}(\mathbb{R}^D \setminus \{\mathbf{0}\}) : [x_1, \dots, x_n] \mapsto (x_1 - x_n, \dots, x_{n-1} - x_n). \quad (2.3)$$

We consider the above isomorphisms (2.2) and (2.3) as identifications. For every finite nonempty subset  $S \subset \mathbb{N}$  we similarly denote:

$$\begin{aligned} \mathbf{E}_S &:= (\mathbb{R}^D)^S / \mathbb{R}^D, \\ \mathbf{F}_S &:= \{(x_j)_{j \in S} \in (\mathbb{R}^D)^S : x_j \neq x_k \text{ for } j \neq k\} / \mathbb{R}^D \quad (\equiv F_S(\mathbb{R}^D) / \mathbb{R}^D). \end{aligned}$$

Now we assume that we are given a differential subalgebra (over the ground field  $\mathbb{k}$ ):

$$\mathcal{O} \equiv \mathcal{O}_2 \subseteq \mathcal{C}^\infty(\mathbb{R}^D \setminus \{\mathbf{0}\}).$$

of the algebra of smooth functions on  $\mathbb{R}^D \setminus \{\mathbf{0}\}$  ( $\equiv \mathbf{F}_2$ ). (As we mentioned, one should think of the algebra  $\mathcal{O}$  as an algebra of “propagators”.) We define several embeddings:

$$\hat{l}_{jk} : \mathcal{O} \hookrightarrow \mathcal{C}^\infty(\mathbf{F}_n) : G(x) \mapsto G(x_j - x_k), \quad 1 \leq j < k \leq n,$$



where (as well as further) **we shall identify the functions belonging to  $\mathcal{C}^\infty(\mathbf{F}_n)$  with translation invariant functions over  $F_n(\mathbb{R}^D)$**  (as in Eq. (2.3)). Then for every  $S \subseteq \{1, \dots, n\}$  with  $|S| \geq 2$  we set

$$\begin{aligned} \mathcal{O}_S &:= \text{the subalgebra of } \mathcal{C}^\infty(\mathbf{F}_n) \text{ generated by all} \\ &\quad \hat{\iota}_{jk}(\mathcal{O}) \text{ for } j, k \in S, j < k, \\ \mathcal{O}_n &:= \mathcal{O}_{\{1, \dots, n\}}. \end{aligned}$$

In fact, we shall not keep  $n$  fixed and we consider the natural inductive limit  $\mathcal{O}_\infty := \bigcup_{n=2}^\infty \mathcal{O}_n$ , and so, for every finite subset  $S \subset \mathbb{N}$  with  $|S| \geq 2$  we consider  $\mathcal{O}_S$  as a subalgebra of  $\mathcal{O}_\infty$ . Note that the algebra  $\mathcal{O}_S$  is linearly spanned by functions of the form

$$G = \prod_{\substack{j, k \in S \\ j < k}} \hat{\iota}_{jk} G_{jk} \equiv \prod_{\substack{j, k \in S \\ j < k}} G_{jk}(x_j - x_k), \quad G_{jk} \in \mathcal{O}. \quad (2.4)$$

Basic examples for the algebra  $\mathcal{O}$  are the algebras  $\kappa[x, 1/x^2]$  and  $\kappa[x, 1/x^2, \log x^2]$ . In the first case the corresponding algebras  $\mathcal{O}_n$  are:

$$\mathcal{O}_n = \kappa[x_1, \dots, x_{n-1}] \left[ \left( \prod_{k=1}^{n-1} x_k^2 \right)^{-1} \left( \prod_{1 \leq j < k \leq n-1} (x_j - x_k)^2 \right)^{-1} \right] \quad (2.5)$$

(under the identification (2.3)). These are the algebras that we need if we perturb massless free fields. From the point of the algebraic geometry the algebra  $\mathcal{O}_n$  coincides exactly with the ring of regular functions on the affine manifold that is complement of union of the quadrics  $x_k^2 = 0$  and  $(x_j - x_k)^2 = 0$ .

We introduce similar notations for the distributions spaces:

$$\begin{aligned} \mathcal{D}'_n &:= \mathcal{D}'(\mathbf{E}_n), \\ \mathcal{D}'_S &:= \{u \in \mathcal{D}'_\infty : u \text{ depends at most on } x_j \text{ for all } j \in S\} \equiv \mathcal{D}'(\mathbf{E}_S), \\ \mathcal{D}'_{S,0} &:= \{u \in \mathcal{D}'_S : \text{supp } u \subseteq \{0\} \subset \mathbf{E}_S\}. \end{aligned}$$

We note that for every inclusion  $S' \subseteq S$  we have natural embeddings  $\mathcal{O}_{S'} \hookrightarrow \mathcal{O}_S$  and  $\mathcal{D}'_{S'} \hookrightarrow \mathcal{D}'_S$ .

## 2.2 Filtrations

A very important notion in the Epstein–Glaser approach to the renormalization is the Steinmann *scaling degree* ([17, Ch. 5]). It corresponds to the

degree of divergence in the other approaches and introduces filtrations on our function spaces.

First, for a distribution  $u \in \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$  the scaling degree is defined as:

$$\text{sc.d. } u := \inf \left\{ \lambda \in \mathbb{R} : \text{w-lim}_{\varepsilon \downarrow 0} \varepsilon^\lambda u(\varepsilon \mathbf{x}) = 0 \right\}$$

(w-lim standing for the weak limit). If  $u \in \mathcal{D}'(\mathbb{R}^N)$  then  $\text{sc.d. } u$  is defined similarly but note that:

$$\text{sc.d. } (u|_{\mathbb{R}^N \setminus \{\mathbf{0}\}}) \leq \text{sc.d. } u \quad (2.6)$$

(for instance, take  $u(\mathbf{x}) = \delta(\mathbf{x})$ ). There is a theorem ([17, Lemma 5.1]) stating that *every distribution belonging to  $\mathcal{D}'(\mathbb{R}^N)$  has a finite scaling degree*. Furthermore, *a necessary and sufficient condition for a distribution on  $\mathbb{R}^N \setminus \{\mathbf{0}\}$  to posses an extension over the whole space  $\mathbb{R}^N$  is the finiteness of its scaling degree* (cf. Lemma 2.5 and the construction after it). Let us also point out the inequalities

$$\text{sc.d. } x^\xi u(\mathbf{x}) \leq -1 + \text{sc.d. } u, \quad \text{sc.d. } \partial_{x^\xi} u(\mathbf{x}) \leq 1 + \text{sc.d. } u \quad (2.7)$$

( $u \in \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ ,  $\xi = 1, \dots, N$ ).

Thus, we obtain *increasing filtrations* on all  $\mathcal{D}'(\mathbb{R}^N)$ ,  $N \in \mathbb{N}$  (and hence, on each  $\mathcal{D}'_S$ ):

$$\begin{aligned} \mathcal{F}_\ell \mathcal{D}'(\mathbb{R}^N) &:= \{u \in \mathcal{D}'(\mathbb{R}^N) : \text{sc.d. } u \leq \ell\}, \quad \ell \in \mathbb{R}, \\ \mathcal{F}_\ell \mathcal{D}'(\mathbb{R}^N) &\subseteq \mathcal{F}_{\ell'} \mathcal{D}'(\mathbb{R}^N) \quad \text{for } \ell \leq \ell', \quad \mathcal{D}'(\mathbb{R}^N) = \bigcup_{\ell \in \mathbb{R}} \mathcal{F}_\ell \mathcal{D}'(\mathbb{R}^N). \end{aligned} \quad (2.8)$$

We introduce the distribution spaces:

$$\mathcal{D}'_{\text{temp}}(\mathbb{R}^N \setminus \{\mathbf{0}\}) := \{u \in \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\}) : \text{sc.d. } u < \infty\}.$$

Then on  $\mathcal{D}'_{\text{temp}}(\mathbb{R}^N \setminus \{\mathbf{0}\})$  we have again a filtration:

$$\mathcal{F}_\ell \mathcal{D}'_{\text{temp}}(\mathbb{R}^N \setminus \{\mathbf{0}\}) := \{u \in \mathcal{D}'_{\text{temp}}(\mathbb{R}^N \setminus \{\mathbf{0}\}) : \text{sc.d. } u \leq \ell\} \quad (\ell \in \mathbb{R}).$$

According to the result mentioned above: *a distribution on  $\mathbb{R}^N \setminus \{\mathbf{0}\}$  belongs to the space  $\mathcal{D}'_{\text{temp}}(\mathbb{R}^N \setminus \{\mathbf{0}\})$  iff it is a restriction of a distribution on  $\mathbb{R}^N$ .*

We also introduce filtrations on the spaces  $\mathcal{O}_S$  (for finite subsets  $S \subset \mathbb{N}$  with  $|S| \geq 2$ ). Starting with  $\mathcal{O} \equiv \mathcal{O}_2 \subseteq \mathcal{C}^\infty(\mathbb{R}^D \setminus \{\mathbf{0}\})$  we introduce first

another, stronger notion of scaling degree on  $\mathcal{C}(\mathbb{R}^N \setminus \{\mathbf{0}\})$  ( $N \in \mathbb{N}$ ): for  $G \in \mathcal{C}^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$  we set

$$\text{Sc.d. } G := \inf \left\{ \lambda \in \mathbb{R} : \exists C_\lambda, R > 0 \text{ such that } |\partial_{\mathbf{x}}^{\mathbf{r}} G(\mathbf{x})| < C_{\lambda, \mathbf{r}} |\mathbf{x}|^{-\lambda - |\mathbf{r}|} \right. \\ \left. \text{for all } \mathbf{r} \in \mathbb{N}_0^N \text{ and } |\mathbf{x}| < R \right\}$$

( $|\mathbf{x}| := \sqrt{\mathbf{x}^2}$ ). Note that for every  $G \in \mathcal{C}^\infty(\mathbb{R}^N \setminus \{\mathbf{0}\})$  we have

$$\text{sc.d. } G \leq \text{Sc.d. } G, \\ \text{Sc.d. } x^\varepsilon G(\mathbf{x}) \leq -1 + \text{Sc.d. } G, \quad \text{sc.d. } \partial_{x^\varepsilon} G(\mathbf{x}) \leq 1 + \text{Sc.d. } G, \quad (2.9)$$

where in the first inequality we view  $G$  also as a distribution over  $\mathbb{R}^N \setminus \{\mathbf{0}\}$ . Then we assume that:

$$\mathcal{O} \subset \mathcal{C}_{temp}^\infty(\mathbb{R}^D \setminus \{\mathbf{0}\}) := \{G \in \mathcal{C}^\infty(\mathbb{R}^D \setminus \{\mathbf{0}\}) : \text{Sc.d. } G < \infty\}$$

and hence, we obtain an increasing filtration on  $\mathcal{O}$ :

$$\mathcal{F}_\ell \mathcal{O} := \{G \in \mathcal{O} : \text{Sc.d. } u \leq \ell\} \quad (\ell \in \mathbb{R}), \\ \mathcal{F}_\ell \mathcal{O} \subseteq \mathcal{F}_{\ell'} \mathcal{O} \quad \text{for } \ell \leq \ell', \quad \mathcal{O} = \bigcup_{\ell \in \mathbb{R}} \mathcal{F}_\ell \mathcal{O}.$$

Now on every  $\mathcal{O}_S$  with  $|S| \geq 3$  we introduce a filtration by the “power counting procedure”. For a general element  $G \in \mathcal{O}_S$ :

$$G = \sum_{\alpha} G_{\alpha}, \quad G_{\alpha} = \prod_{\substack{j, k \in S \\ j < k}} \hat{t}_{jk} G_{jk}^{\alpha}, \quad G_{jk}^{\alpha} \in \mathcal{O} \quad (2.10)$$

we set a **degree of divergence**

$$\text{dev.d. } G := \inf_{\substack{\text{all possible rep-} \\ \text{resentations (2.10)}}} \left\{ \max_{\alpha} \left\{ \sum_{\substack{j, k \in S \\ j < k}} \text{Sc.d. } G_{jk}^{\alpha} \right\} \right\},$$

Thus, we obtain an increasing filtration:

$$\mathcal{F}_\ell \mathcal{O}_S := \{G \in \mathcal{O}_S : \text{dev.d. } G \leq \ell\} \quad (\ell \in \mathbb{R}), \quad (2.11)$$

$$\mathcal{F}_\ell \mathcal{O}_S \subset \mathcal{F}_{\ell'} \mathcal{O}_S \quad \text{for } \ell \leq \ell', \quad \mathcal{O}_S = \bigcup_{\ell \in \mathbb{R}} \mathcal{F}_\ell \mathcal{O}_S.$$

In the sequel we shall need the following statement

**Lemma 2.1.** (a) Let  $u \in \mathcal{D}'_{temp}(R^N \setminus \{0\})$  and  $F \in \mathcal{C}^\infty_{temp}(R^N \setminus \{0\})$ , then we have:  $\text{sc.d.}(Fu) \leq \text{Sc.d.} F + \text{sc.d.} u$ .

(b) Let  $u_k \in \mathcal{D}'_{temp}(R^{N_k} \setminus \{0\})$  for  $k = 1, \dots, m$ . Then  $u = u_1 \otimes \dots \otimes u_m \in \mathcal{D}'_{temp}(R^N \setminus \{0\})$ ,  $N = N_1 + \dots + N_m$  and  $\text{sc.d.} u = \text{sc.d.} u_1 + \dots + \text{sc.d.} u_m$ .

*Sketch of the proof.* (a) By the Banach-Steinhaus theorem ([16]) it follows that:  $\text{sc.d.} u < \lambda$  iff for every compact  $K \subset R^N \setminus \{0\}$  there exist  $L = L(\lambda) \in \mathbb{N}_0$ , a test functions norm  $\|\cdot\|_{K,L}$ ,

$$\|f\|_{K,L} := \sup_{\mathbf{x} \in K, |\mathbf{r}| \leq L} |\partial_{\mathbf{x}}^{\mathbf{r}} f(\mathbf{x})|,$$

and a constant  $C_{K,\lambda} > 0$  such that for every  $f \in \mathcal{D}(R^N)$  with  $\text{supp } f \subseteq K$  and  $\varepsilon \in (0, 1)$  we have:

$$|u[f(\varepsilon^{-1}\mathbf{x})]| \leq C_{K,\lambda} \|f\|_{K,L} \varepsilon^{N-\lambda}.$$

Let  $\text{Sc.d.} F + \text{sc.d.} u < \lambda$ . There exist  $\lambda_1, \lambda_2$  such that  $\text{Sc.d.} F < \lambda_1$ ,  $\text{sc.d.} u < \lambda_2$  and  $\lambda = \lambda_1 + \lambda_2$ . Hence,

$$\begin{aligned} |(Fu)[f(\varepsilon^{-1}\mathbf{x})]| &\leq C_{K,\lambda_2} \|F(\varepsilon\mathbf{x}) f(\mathbf{x})\|_{K,L_2} \varepsilon^{N-\lambda_2} \\ &\leq C' \|f(\mathbf{x})\|_{K,L_2} \varepsilon^{N-\lambda_1-\lambda_2}, \end{aligned}$$

where  $L_2 = L(\lambda_2)$ . It follows that  $\text{sc.d.}(Fu) < \lambda$ .

(b) We have:  $\text{sc.d.} u < \lambda$  iff for every compact  $K \subset R^N$  we have inequality  $|u[f_1(\varepsilon^{-1}\mathbf{x}_1) \dots f_m(\varepsilon^{-1}\mathbf{x}_m)]| \leq C \|f_1\|_{K,L_1} \dots \|f_m\|_{K,L_m} \varepsilon^{N-\lambda}$ . Then by the kernel and the Banach-Steinhaus theorems we obtain an inequality  $|u[f(\varepsilon^{-1}\mathbf{x})]| \leq C' \|f\|_{K,L} \varepsilon^{N-\lambda}$  every  $f \in \mathcal{D}(R^N \setminus \{0\})$ .  $\square$

### 2.3 Preliminary definition of renormalization maps

We proceed to give a preliminary definition of renormalization maps. It will be just a list of desirable properties for them.

A system renormalization maps is a collection of linear maps:

$$R_S : \mathcal{O}_S \rightarrow \mathcal{D}'_S \quad (R_{\{1,\dots,n\}} =: R_n), \quad (2.12)$$

indexed by all finite subsets  $S \subset \mathbb{N}$  with at least two elements. They are assumed to satisfy the properties (r1)–(r4) listed below. Let us stress that

we do not assume any nontrivial continuity of the maps  $R_S$  (2.12) and we consider them just as linear maps.

(r1) For every bijection  $\sigma : S \cong S'$  we require:

$$\sigma^* \circ R_{S'} = R_S \circ \sigma^*,$$

where  $(\sigma^* F)(x_{j_1}, \dots, x_{j_n}) := F(x_{\sigma(j_1)}, \dots, x_{\sigma(j_n)})$  for a function or distribution  $F$ .

Thus, by (r1) all  $R_S$  are characterized by  $R_n$  for  $n = |S| = 2, 3, \dots$

(r2) Every  $R_S$  preserves the filtrations:  $R_S \mathcal{F}_\ell \mathcal{O}_S \subseteq \mathcal{F}_\ell \mathcal{D}'_S$  ( $\ell \in \mathbb{R}$ ). In other words, we require for  $|S| \geq 3$ :

$$\text{sc.d. } R_S G \leq \text{dev.d. } G, \quad (2.13)$$

which is also true for  $|S| = 2$  if we set then  $\text{dev.d. } G \equiv \text{Sc.d. } G$ .

(r3) For every polynomial  $f \in \kappa[\mathbf{E}_n]$  and  $G \in \mathcal{O}_n$  we have:

$$R_n f G = f R_n G.$$

All the above conditions are of linear type with respect to  $\{R_n\}$ , but the last one is “nonlinear”. To state it we need more notations. Let  $\mathfrak{P} = \{S_1, \dots, S_k\}$  be an  $S$ -partition, i.e.,  $S = S_1 \dot{\cup} \dots \dot{\cup} S_k$  for nonempty  $S_1, \dots, S_k$  (in the case  $S = \{1, \dots, n\}$  we shall say  $n$ -partition). The  $S$ -partition  $\mathfrak{P}$  can be characterized also as an equivalence relation on  $S$  with equivalence classes  $S_1, \dots, S_k$ . This relation we denote by  $\sim_{\mathfrak{P}}$ . Then, by the construction of the algebra  $\mathcal{O}_S$  it is linearly spanned by elements of a form:

$$G_S = G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} G_{S'}, \quad (2.14)$$

where  $G_{S'} \in \mathcal{O}_{S'}$  for  $S' \in \mathfrak{P}$  and  $G_{\mathfrak{P}} \in \mathcal{O}_{\mathfrak{P}}$ ,

$$\mathcal{O}_{\mathfrak{P}} := \text{the subalgebra of } \mathcal{C}^\infty(\mathbf{F}_n) \text{ generated by all } \hat{\iota}_{jk}(\mathcal{O}) \text{ for } j, k \in S, j \sim_{\mathfrak{P}} k. \quad (2.15)$$

In the case when  $\mathfrak{P}$  contains  $S'$  with  $|S'| = 1$  we shall assume in Eq. (2.14) that  $G_{S'} := 1$  and set also

$$\mathcal{O}_{S'} = \mathcal{O}_1 := \kappa \quad (|S'| = 1).$$

Another extreme case is when  $|\mathfrak{P}| = 1$  (i.e.,  $\mathfrak{P} = \{S\}$ ) and then we shall again assume  $G_{\mathfrak{P}} := 1$  and set  $\mathcal{O}_{\mathfrak{P}} := \mathcal{K}$ . Finally, we define the following open subsets of  $\mathbf{E}_n$

$$\mathbf{F}_{\mathfrak{P}} = \{[x_1, \dots, x_n] \in \mathbf{E}_n : x_j \neq x_k \text{ if } j \approx_{\mathfrak{P}} k\} \quad (2.16)$$

(this is for the case of an  $n$ -partition  $\mathfrak{P}$  and similarly one introduces  $\mathbf{F}_{\mathfrak{P}}$  for arbitrary  $S$ -partition). Note that for the case  $|\mathfrak{P}| = 1$  Eq. (2.15) becomes an identity and  $\mathbf{F}_{\mathfrak{P}} = \emptyset$ . Partitions containing at least two elements are called **proper**. Let us also point out the similarity between the definition of  $\mathbf{F}_{\mathfrak{P}}$  and the definition of the configuration spaces  $\mathbf{F}_n$ . In fact,  $\mathbf{F}_{\{\{1\}, \dots, \{n\}\}} \equiv \mathbf{F}_n$ . Because of this similarity we have chosen one and the same letter for the notations. In the case of algebras  $\mathcal{O}_n$  (2.5) the algebra  $\mathcal{O}_{\mathfrak{P}}$  contains exactly those the elements of  $\mathcal{O}_n$ , which are regular functions on  $\mathbf{F}_{\mathfrak{P}}$  (similarly,  $\mathcal{O}_n$  contains regular functions on  $\mathbf{F}_n$ ).

(r4) For every *proper*  $S$ -partition  $\mathfrak{P}$  we have:

$$R_S G_S \Big|_{\mathbf{F}_{\mathfrak{P}}} = G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} R_{S'} G_{S'} \quad (2.17)$$

where  $G_S$  is of the form (2.14).

In the extreme case of (r4) when  $\mathfrak{P}$  contains  $S'$  with  $|S'| = 1$  we set in addition to the above convention  $\mathcal{O}_1 := \mathcal{K}$  that:

$$R_1 := \text{id}.$$

Then we obtain as a consequence that

$$R_n G \Big|_{\mathbf{F}_n} = G$$

for all  $G \in \mathcal{O}_n$  (since  $\mathbf{F}_{\{\{1\}, \dots, \{n\}\}} = \mathbf{F}_n$ ).

Another corollary of (r4) is that if we have a system of linear maps  $R_{S'}$  (2.12) that are defined for all finite  $S' \subset \mathbb{N}$  with  $2 \leq |S'| \leq n-1$  and satisfy (r1)–(r4) then we have linear maps for all  $S$  with  $|S| = n$ :

$$\dot{R}_S : \mathcal{O}_S \rightarrow \mathcal{D}'(\mathbf{E}_S \setminus \{\mathbf{0}\})$$

uniquely determined by the condition that  $\dot{R}_S G_S \Big|_{\mathbf{F}_{\mathfrak{P}}}$  is equal to the right hand side of Eq. (2.17) for every proper  $S$ -partition  $\mathfrak{P}$ . This follows from the fact that  $\mathbf{F}_{\mathfrak{P}}$  form an open covering of  $\mathbf{E}_S \setminus \{\mathbf{0}\}$ :

$$\mathbf{E}_S \setminus \{\mathbf{0}\} = \bigcup_{\substack{\mathfrak{P} \text{ is a proper} \\ S\text{-partition}}} \mathbf{F}_{\mathfrak{P}}. \quad (2.18)$$

Furthermore, if the linear maps  $R_{S'}$  ( $|S'| \leq n - 1$ ) are part of a complete system of renormalization maps then

$$R_S G \Big|_{\mathbf{E}_S \setminus \{\mathbf{0}\}} = \dot{R}_S G \quad (2.19)$$

for all  $G \in \mathcal{O}_S$ . We set again

$$\dot{R}_n := \dot{R}_{\{1, \dots, n\}},$$

and clearly, all  $\dot{R}_S$  with  $|S| = n$  are isomorphic to  $\dot{R}_n$ .

**Lemma 2.2.** *The image of the linear map  $\dot{R}_n$  is contained in the space  $\mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\})$ . In fact,  $\dot{R}_n$  preserves the filtrations:*

$$\dot{R}_n \mathcal{F}_\ell \mathcal{O}_n \subseteq \mathcal{F}_\ell \mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\}) \quad (\ell \in \mathbb{R}). \quad (2.20)$$

It is enough to *prove* Eq. (2.20). We do this by induction in  $n = 2, 3, \dots$ . For  $n = 2$  Eq. (2.20) follows by definition: here  $\dot{R}_2 G = R_2 G \Big|_{\mathbb{R}^D \setminus \{\mathbf{0}\}} = G$  for  $G \in \mathcal{O}_2$ , i.e.,  $\dot{R}_2$  is the inclusion  $\mathcal{O}_2 \hookrightarrow \mathcal{D}'(\mathbf{E}_2 \setminus \{\mathbf{0}\})$ , and then we apply the first of Eqs. (2.9). For  $n > 2$  we have to prove that the inequality  $\text{sc.d.} \dot{R}_n G \leq \text{dev.d.} G$  (i.e., Eq. (2.13)) holds for every  $G \in \mathcal{O}_n$ .

To this end we first note that the notions of scaling degrees sc.d. (resp., Sc.d.) can be also introduced for distributions (resp., smooth functions) over open cones  $\mathcal{U} \subseteq \mathbb{R}^N \setminus \{\mathbf{0}\}$ . Then Lemma 2.1 (a) holds also for  $u \in \mathcal{D}'_{temp}(\mathcal{U})$  and  $F \in \mathcal{C}^\infty_{temp}(\mathcal{U})$ . Note that  $\mathbf{F}_{\mathfrak{P}}$  are open cones in  $\mathbf{E}_n \setminus \{\mathbf{0}\}$  for proper  $n$ -partitions  $\mathfrak{P}$  and

$$\text{sc.d.} u = \max_{\substack{\mathfrak{P} \text{ is a proper} \\ S\text{-partition}}} \text{sc.d.} \left( u \Big|_{\mathbf{F}_{\mathfrak{P}}} \right)$$

(this can be proven by using a partition of unity, which is subordinate to the open covering  $\{\mathbf{F}_{\mathfrak{P}}\}$ ).

Thus, let us fix an arbitrary proper  $n$ -partition  $\mathfrak{P}$  and set  $S := \{1, \dots, n\}$ . By the construction of the filtration on  $\mathcal{O}_n$ , the function  $G$  has a representation of the form (2.10), such that for all  $\alpha$ :  $\sum_{1 \leq j < k \leq n} \text{Sc.d.} G_{jk}^\alpha \leq \text{dev.d.} G$ . Then every  $G_\alpha$  in (2.10) has a representation of a type (2.14):  $G_\alpha = G_{\mathfrak{P}}^\alpha \cdot \prod_{S' \in \mathfrak{P}} G_{S'}^\alpha$ . Note that  $G_{\mathfrak{P}}^\alpha \Big|_{\mathbf{F}_{\mathfrak{P}}} \in \mathcal{C}^\infty_{temp}(\mathbf{F}_{\mathfrak{P}})$  and  $\text{Sc.d.} \left( G_{\mathfrak{P}}^\alpha \Big|_{\mathbf{F}_{\mathfrak{P}}} \right) \leq$

$\sum_{\substack{j,k \in S \\ j \approx_{\mathfrak{P}} k}} \text{Sc.d. } G_{jk}^\alpha$ . Then we have  $\text{Sc.d. } (G_{\mathfrak{P}}^\alpha |_{\mathbf{F}_{\mathfrak{P}}}) + \sum_{S' \in \mathfrak{P}} \text{dev.d. } G_{S'}^\alpha \leq \text{dev.d. } G$

for every  $\alpha$ . Applying the inductive assumption we obtain that

$\text{Sc.d. } (G_{\mathfrak{P}}^\alpha |_{\mathbf{F}_{\mathfrak{P}}}) + \sum_{S' \in \mathfrak{P}} \text{sc.d. } R_{S'} G_{S'}^\alpha \leq \text{dev.d. } G$ . Then by Lemma 2.1 (a)

and (b) (over  $\mathbf{F}_{\mathfrak{P}}$ ) and conclude that  $\text{sc.d. } (\dot{R}_n G |_{\mathbf{F}_{\mathfrak{P}}}) \leq \text{dev.d. } G$ . Since this

is true for every proper  $n$ -partition  $\mathfrak{P}$  it follows that  $\text{sc.d. } \dot{R}_n G \leq \text{dev.d. } G$ .

This completes the proof of Lemma 2.2.

Thus, we shall consider the maps  $\dot{R}_n$  as linear maps

$$\dot{R}_n : \mathcal{O}_n \rightarrow \mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\}). \quad (2.21)$$

**Remark 2.1.** One can enhance the condition (r1) to a stronger condition:

$$(\hat{\iota})^* \circ R_{S'} = R_S \circ (\hat{\iota})^*$$

for every *injection*  $\hat{\iota} : S' \rightarrow S$ . In particular,  $R_S |_{\mathcal{O}_{S'}} = R_{S'}$  if  $S' \subseteq S$ , the restriction being taken under the inclusion  $\mathcal{O}_{S'} \subseteq \mathcal{O}_S$ . This is quite natural assumption but since we shall not use it here we do not impose it. (See also Remarks 2.2 and 2.3.)

## 2.4 Primary renormalization maps and definition of renormalization maps

Equation (2.19) suggests to built the renormalization maps  $R_n$  as compositions of the recursively defined linear maps  $\dot{R}_n$  and extension maps  $\mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'(\mathbf{E}_n)$ . These maps we call *primary renormalization maps* and using them we shall define  $R_n$ .

Thus, a **system of primary renormalization maps** is a collection of linear maps

$$\mathcal{P}_N : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'(\mathbb{R}^N) \quad (2.22)$$

labelled by integers  $N = 1, 2, \dots$  and satisfying properties (p1)–(p5) listed below. As in the case of the renormalization maps  $R_n$ , we emphasize that no continuity is assumed for the maps  $\mathcal{P}_N$  either.

(p1) Extension property:

$$(\mathcal{P}_N u) \Big|_{\mathbb{R}^N \setminus \{\mathbf{0}\}} = u \quad (u \in \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\})).$$



(p2) Preservation of filtrations:

$$\mathcal{P}_N \mathcal{F}_\ell \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \subseteq \mathcal{F}_\ell \mathcal{D}'(\mathbb{R}^N) \quad (N = 1, 2, \dots, \ell \in \mathbb{R}).$$

(p3) Orthogonal invariance:

$$O^* \circ \mathcal{P}_N \circ (O^*)^{-1} = \mathcal{P}_N,$$

for every orthogonal transformation  $O$  of  $\mathbb{R}^N$ , where  $(O^*F)(\mathbf{x}) := F(O\mathbf{x})$  for a function (distribution)  $F(\mathbf{x})$  over  $\mathbb{R}^N$ .

(p4) For every polynomial  $f \in \kappa[\mathbb{R}^N]$  and  $u \in \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\})$ :

$$\mathcal{P}_N f u = f \mathcal{P}_N u.$$

(p5) If  $u(\mathbf{x}) \in \mathcal{D}'(\mathbb{R}^M)$  with  $\text{supp } u_1 \subseteq \{\mathbf{0}\}$  and  $v(\mathbf{y}) \in \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ , where  $M, N \geq 1$ , then:

$$\mathcal{P}_{M+N}(u \otimes v) = u \otimes \mathcal{P}_N(v)$$

$$((u \otimes v)(\mathbf{x}, \mathbf{y}) := u(\mathbf{x}) v(\mathbf{y})).$$

This completes the definition of primary renormalization maps.

Let us point out the following stronger version of (p5), which under condition (p3) is equivalent to (p5):

(p5') For every (orthogonal) decomposition  $\mathbb{R}^{M+N} = \mathbf{V} \oplus \mathbf{V}'$ ,  $\mathbf{V} \cong \mathbb{R}^M$  and  $\mathbf{V}' \cong \mathbb{R}^N$  with  $M, N \geq 1$ : if  $u(\mathbf{x}) \in \mathcal{D}'(\mathbf{V})$  with  $\text{supp } u_1 \subseteq \{\mathbf{0}\}$  and  $v(\mathbf{y}) \in \mathcal{D}'(\mathbf{V}' \setminus \{\mathbf{0}\})$  then  $\mathcal{P}_{M+N}(u \otimes v) = u \otimes \mathcal{P}_N(v)$ .

We shall need only the renormalization maps over  $\mathbb{R}^{D(n-1)} \cong \mathbf{E}_n$ , where the identification (2.2) is assumed and we shall also lift, by the Euclidean invariance, these maps from  $\mathbf{E}_n$  to all  $\mathbf{E}_S$  for finite subsets  $S \subset \mathbb{N}$ . Let us denote the resulting linear maps by:

$$P_S (\cong \mathcal{P}_{D(n-1)}) : \mathcal{D}'_{temp}(\mathbf{E}_S \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'(\mathbf{E}_S), \quad P_n := P_{\{1, \dots, n\}}. \quad (2.23)$$

Now the construction and together, the definition of renormalization maps is based on the following theorem:

**Theorem 2.3.** *Let we be given by a system of primary renormalization maps  $\{P_n\}_{n=2}^\infty$  and define recursively:*

$$R_2 := P_2, \quad (2.24)$$

$$R_n := \mathcal{P}_n \circ \dot{R}_n \quad \text{for } n > 2. \quad (2.25)$$

Here: having defined  $R_n$  by the induction we then define  $R_S$  for all  $S \subset \mathbb{N}$  with  $|S| = n$  by using the permutation symmetry and the orthogonal invariance implied by (p1) and (p3); and finally, we define  $\dot{R}_n$  provided that  $R_S$  satisfy (r1)–(r4) for  $|S| < n$ . In this way we obtain a system of linear maps  $\{R_S\}_S$  satisfying (r1)–(r4). The so defined  $R_n$  (or,  $R_S$ ) we shall call **renormalization maps**.

*Proof.* We use an induction in  $n = 2, 3, \dots$ . The most nontrivial part in the proof is contained in Lemma 2.2 that ensures property (r2). Property (r3) is a consequence of (p4). Condition (r4) is satisfied due to the construction of  $\dot{R}_n$ . The permutation symmetry required in (r1) follows from (p1) and the restriction property in (r1) follows by induction.  $\square$

**Remark 2.2.** If we had imposed for the renormalization maps instead of condition (r1) its stronger version in Remark 2.1, then we would also need a stronger version for condition (p3). Namely, for every partial isometry  $O : \mathbb{R}^N \rightarrow \mathbb{R}^M$  we should impose

$$O^* \circ \mathcal{P}_N = \mathcal{P}_M \circ O^*,$$

where for a distribution  $F(\mathbf{x})$  over  $\mathbb{R}^M$ ,  $(O^*F)(\mathbf{x}) := F(O\mathbf{x})$  is a distribution over  $\mathbb{R}^N$ .

## 2.5 Construction of primary renormalization maps

In this subsection we shall prove the existence of primary renormalization maps. As a more technical section it can be skipped on the first reading.

**Theorem 2.4.** *There exists a system of primary renormalization maps.*

Part of this theorem is based on the old results of Epstein–Glaser and Steinmann on renormalization and the new element here is mainly to achieve properties (p4) and (p5). Thus, we begin by stating a known result:

**Lemma 2.5.** *For every  $N = 1, 2, \dots$  there exists a unique linear map*

$$\mathcal{P}_{N,0} : \mathcal{F}_N \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathcal{F}_N \mathcal{D}'(\mathbb{R}^N)$$

*such that  $u = \mathcal{P}_{N,0}u|_{\mathbb{R}^N \setminus \{\mathbf{0}\}}$  for every  $u \in \mathcal{F}_N \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ . It has also the property: sc.d.  $\mathcal{P}_{N,0}u =$  sc.d.  $u$ .*

The *proof* of Lemma 2.5 can be found in [3, Theorem 2].

Continuing with the *proof of Theorem 2.4* we first define linear maps  $\mathcal{P}'_N$  that fulfill just properties (p1) and (p2). (They would be extensions of the corresponding  $\mathcal{P}_{N,0}$  provided by the above lemma.)

To this end we take a test function  $\vartheta(\mathbf{x}) \in \mathcal{D}(\mathbb{R}^N)$  that is equal to 1 in a neighborhood of 0 and introduce for test functions  $f(\mathbf{x}) \in \mathcal{D}(\mathbb{R}^N)$  the truncated, first order Taylor remainder

$$\sum_{\xi=1}^N x^\xi T_\xi(f)(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{0}) \vartheta(\mathbf{x}),$$

so that  $T_\xi(f) \in \mathcal{D}(\mathbb{R}^N)$ . Then we set inductively for  $\ell = 1, 2, \dots$ :

$$\begin{aligned} \mathcal{P}_{N,\ell} : \mathcal{F}_{N+\ell} \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\}) &\rightarrow \mathcal{F}_{N+\ell} \mathcal{D}'(\mathbb{R}^N), \\ \mathcal{P}_{N,\ell}(u)[f] &:= \sum_{\xi=1}^N \mathcal{P}_{N,\ell-1}(x^\xi u)[T_\xi(f)(\mathbf{x})] \end{aligned} \quad (2.26)$$

( $u \in \mathcal{F}_\ell \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ ), the right hand side being well defined due to the fact that  $x^\xi u \in \mathcal{F}_{N+\ell-1} \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ . The so defined  $\mathcal{P}_{N,\ell}$  then satisfy  $u = \mathcal{P}_{N,\ell} u|_{\mathbb{R}^N \setminus \{\mathbf{0}\}}$

**Lemma 2.6.** *For all  $\ell = 0, 1, \dots$  we have:*

$$\text{sc.d. } \mathcal{P}_{N,\ell} u \leq \text{sc.d. } u \quad (2.27)$$

for every  $u \in \mathcal{F}_{N+\ell} \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ .

*Proof.* For  $\ell = 0$  the statement follows by Lemma 2.5. Assume by induction, we have proven the lemma for  $\ell - 1$  and  $\ell > 0$ . We should prove that if  $\text{sc.d. } u < \lambda$  then for every compact  $K \subset \mathbb{R}^N \setminus \{\mathbf{0}\}$  there exist  $L = L(\lambda) \in \mathbb{N}_0$ , a test functions norm  $\|\cdot\|_{K,L}$ , and a constant  $C_{K,\lambda} > 0$  such that for every  $f \in \mathcal{D}(\mathbb{R}^N)$  with  $\text{supp } f \subseteq K$  and  $\varepsilon \in (0, 1)$  we have:

$$|\mathcal{P}_{N,\ell}(u)[f_\varepsilon]| \leq C_{K,\lambda} \|f\|_{K,L} \varepsilon^{N-\lambda}, \quad (2.28)$$

where  $f_\varepsilon(\mathbf{x}) := f(\varepsilon^{-1}\mathbf{x})$  (as in Lemma 2.1). We prove Eq. (2.28) separately for the case of test functions  $f$  such that  $f(0) = 0$  and the case when  $f = \vartheta$ .

If  $f(0) = 0$  then  $|\mathcal{P}_{N,\ell}(u)[f_\varepsilon]| \leq \sum_{\xi=1}^N |\mathcal{P}_{N,\ell}(x^\xi u)[T_\xi f_\varepsilon]|$ . Since  $T_\xi f_\varepsilon = \varepsilon^{-1}(T_\xi f)_\varepsilon$  and  $\text{sc.d. } \mathcal{P}_{N,\ell-1}(x^\xi u) \leq \text{sc.d. } x^\xi u \leq -1 + \text{sc.d. } u \leq -1 + \lambda$  then using the inductive assumption we obtain the estimate (2.28).

For  $f = \vartheta$  we have  $|\mathcal{P}_{N,\ell}(u)[\vartheta_\varepsilon]| = |u[\vartheta - \vartheta_\varepsilon]|$ . Then setting  $\varepsilon = 2^{-r}$  we get an estimate  $|\mathcal{P}_{N,\ell}(u)[\vartheta_\varepsilon]| \leq \sum_{s=0}^{r-1} |u[\vartheta_{2^{-s}} - \vartheta_{2^{-s-1}}]| = \sum_{s=0}^{r-1} |u[(\vartheta -$

$\vartheta_{2^{-1}})_{2^{-s}} \Big] \Big| \leq C' \sum_{s=0}^{r-1} 2^{-s(N-\lambda)} \leq C'' \varepsilon^{N-\lambda}$  for some positive constants  $C', C''$  provided that  $N \neq \lambda$ , which is not an essential restriction. This completes the proof of the lemma.  $\square$

We note further that:

$$\mathcal{P}_{N,\ell+1} \Big|_{\mathcal{F}_\ell \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})} = \mathcal{P}_{N,\ell} \quad \text{for all } \ell = 1, 2, \dots,$$

since

$$\mathcal{P}_{N,\ell+1} u - \mathcal{P}_{N,\ell} u = -(\mathcal{P}_{N,\ell} u)[\vartheta] \delta(\mathbf{x}),$$

for  $u \in \mathcal{F}_{N+\ell} \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ , but on the other hand,  $(\mathcal{P}_{N,\ell} u)[\vartheta] = 0$  for  $\ell > 0$ , because  $T_\xi(\vartheta) \equiv 0$ . Still, we have an inconsistency:

$$\mathcal{P}_{N,1} \Big|_{\mathcal{F}_N \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})} - \mathcal{P}_{N,0} = \alpha_0(u) \delta(\mathbf{x}),$$

where  $\alpha_0$  is the linear functional:

$$\alpha_0 : u \mapsto -(\mathcal{P}_{N,0} u)[\vartheta] : \mathcal{F}_N \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathbb{R}.$$

Hence, to correct this we have to modify  $\mathcal{P}_{N,1}$  in the following way:

$$\mathcal{P}'_{N,1} u := \mathcal{P}_{N,1} u - \alpha(u) \delta(\mathbf{x})$$

where  $\alpha : \mathcal{F}_{N+1} \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathbb{R}$  is a linear functional that is a continuation of  $\alpha_0$  (such a continuation always exists). Note that  $\mathcal{P}'_{N,1}$  satisfies the inequality (2.28). Then introducing inductively linear maps  $\mathcal{P}'_{N,\ell}$  for  $\ell = 1, 2, \dots$ , again by Eq. (2.26), we obtain a consistent system of linear maps, all satisfying Eq. (2.28) according to Lemma 2.6. Thus, we can define a linear map

$$\mathcal{P}'_N : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'(\mathbb{R}^N)$$

by setting

$$\mathcal{P}'_N \Big|_{\mathcal{F}_{N+\ell} \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})} := \mathcal{P}'_{N,\ell}$$

for  $\ell > 0$ . This map then satisfies the properties (p1) and (p2) as well as,

$$\mathcal{P}'_N \Big|_{\mathcal{F}_N \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})} = \mathcal{P}_{N,0}.$$

Next we consider the problem of fulfilling conditions (p3), (p4) and (p5). Note first that if we find a system of linear maps  $\mathcal{P}_N$  satisfying (p1), (p2), (p4) and (p5') then by averaging over the compact group  $O(N)$ :

$$\int_{O(N)} O^* \circ \mathcal{P}_N \circ (O^*)^{-1} d\mu(O)$$

we will obtain a system of linear maps satisfying all conditions (p1)–(p5).

**Lemma 2.7.** *There exists a linear map  $\mathcal{P}_N''$  that satisfies properties (p1), (p2) and (p4).*

*Proof.* We have to fulfill in addition to (p1) and (p2) the equalities

$$\mathcal{P}_N''(x^\xi u) = x^\xi \mathcal{P}_N''(u) \quad (2.29)$$

for every  $\xi = 1, \dots, N$  (where  $\mathbf{x} = (x^1, \dots, x^N)$ ). By the above considerations there exists a linear map  $\mathcal{P}_N' : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'(\mathbb{R}^N)$  satisfying properties (p1) and (p2). If  $\mathcal{P}_N'' : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'(\mathbb{R}^N)$  is another map that satisfies (p1) and (p2) we set

$$\begin{aligned} \mathfrak{Q} &:= \mathcal{P}_N' - \mathcal{P}_N'' : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'_{\mathbb{R}^N, 0}, \\ c_\xi &:= x^\xi \circ \mathcal{P}_N' - \mathcal{P}_N' \circ x^\xi : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'_{\mathbb{R}^N, 0} \end{aligned} \quad (2.30)$$

( $\xi = 1, \dots, N$ ), where  $\mathcal{D}'_{\mathbb{R}^N, 0}$  stands for the space of distributions on  $\mathbb{R}^M$  supported at zero. Then Eq. (2.29) is equivalent to

$$c_\xi = x^\xi \circ \mathfrak{Q} - \mathfrak{Q} \circ x^\xi. \quad (2.31)$$

Thus, the problem is to find a linear map  $\mathfrak{Q}$  (2.30) such that it preserves the filtrations and (2.31) is satisfied.

To this end we expand  $\mathfrak{Q}$  and  $c_\xi$  in delta functions and their derivatives:

$$\mathfrak{Q} = \sum_{\mathbf{r} \in \mathbb{N}_0^N} \frac{1}{\mathbf{r}!} \delta^{(\mathbf{r})}(\mathbf{x}) Q_{\mathbf{r}}, \quad Q_{\mathbf{r}} : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathbb{R}, \quad (2.32)$$

$$c_\xi = \sum_{\mathbf{r} \in \mathbb{N}_0^N} \frac{1}{\mathbf{r}!} \delta^{(\mathbf{r})}(\mathbf{x}) C_{\xi, \mathbf{r}}, \quad C_{\xi, \mathbf{r}} : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathbb{R} \quad (2.33)$$

(recall the multiindex notations:  $\mathbf{x} = (x^1, \dots, x^N) \in \mathbb{R}^N$ ;  $\mathbf{r} = (r_1, \dots, r_N) \in \mathbb{N}_0^N$ ,  $|\mathbf{r}| = \sum_j r_j$ ,  $\mathbf{r}! = \prod_j r_j!$ ,  $\mathbf{x}^{\mathbf{r}} = \prod_j (x^j)^{r_j}$  and  $\delta^{(\mathbf{r})}(\mathbf{x}) = \partial^{\mathbf{r}} \delta(\mathbf{x})$ ). The condition that  $\mathfrak{Q}$  preserves the filtrations is equivalent to

$$Q_{\mathbf{r}} u = 0 \quad \text{if} \quad |\mathbf{r}| > \text{sc.d. } u - N. \quad (2.34)$$

Equation (2.31) holds iff

$$Q_{\mathbf{r}+\mathbf{e}_\xi}[u] = -C_{\xi,\mathbf{r}}[u] - Q_{\mathbf{r}}[x^\xi u] \quad (2.35)$$

for all  $\mathbf{r} \in \mathbb{N}_0^N$ , where  $\mathbf{e}_\xi$  is the  $\xi$ th basic vector in  $\mathbb{R}^N$  (this follows from the representations (2.33) and (2.32), and the formula  $x^\xi \delta^{(\mathbf{r})}(\mathbf{x}) = -r_\xi \delta^{(\mathbf{r}-\mathbf{e}_\xi)}(\mathbf{x})$ ). So, we have to find a collection of linear functionals  $Q_{\mathbf{r}}$ , which satisfy Eqs. (2.34) and (2.35).

The linear maps  $c_\xi$  satisfy an “integrability” relation

$$c_\xi \circ x^\eta - x^\xi \circ c_\eta = c_\eta \circ x^\xi - x^\eta \circ c_\xi,$$

which implies

$$C_{\xi,\mathbf{r}+\mathbf{e}_\eta}[u] - C_{\xi,\mathbf{r}}[x^\eta u] = C_{\eta,\mathbf{r}+\mathbf{e}_\xi}[u] - C_{\eta,\mathbf{r}}[x^\xi u]. \quad (2.36)$$

By the fact that  $\mathcal{P}'_N$  preserves the filtrations and Eq. (2.7) we obtain:

$$C_{\xi,\mathbf{r}} u = 0 \quad \text{if} \quad |\mathbf{r}| > \text{sc.d. } u - 1 - N. \quad (2.37)$$

Then let us set

$$Q_{\mathbf{r}} u := \sum_{\xi=1}^N \sum_{s=1}^{r_\xi} (-1)^{|\mathbf{q}(\xi,s)|} C_{\xi,\mathbf{r}-\mathbf{q}(\xi,s)} [\mathbf{x}^{\mathbf{q}(\xi,s)-\mathbf{e}_\xi} u], \quad (2.38)$$

where  $\mathbf{q}(\xi,s) := s \mathbf{e}_\xi + \sum_{\eta=1}^{\xi-1} r_\eta \mathbf{e}_\eta$  (writing a sum  $\left( \sum_{j=a}^b \dots \right)$  with  $a, b \in \mathbb{Z}$  we set it zero if  $a > b$ ). Note that the so defined  $Q_{\mathbf{r}}$  satisfy condition (2.34) since Eq. (2.37) implies that  $C_{\xi,\mathbf{r}-\mathbf{q}(\xi,s)} [\mathbf{x}^{\mathbf{q}(\xi,s)-\mathbf{e}_\xi} u] = 0$  if

$$|\mathbf{r} - \mathbf{q}(\xi,s)| > \text{sc.d. } (\mathbf{x}^{\mathbf{q}(\xi,s)-\mathbf{e}_\xi} u) - 1 - N \iff |\mathbf{r}| > \text{sc.d. } u - N$$

Equation (2.35) is also satisfied, because of (2.36). Thus,  $Q_{\mathbf{r}}$  (2.38) determine a linear map  $\mathfrak{Q}$  such that Eq. (2.31) holds and then  $\mathcal{P}''_N := \mathcal{P}'_N - \mathfrak{Q}$  fulfills the conditions of the lemma.  $\square$

To complete the proof of Theorem 2.4 it remains to fulfill condition (p5). In fact, we have to fulfill its stronger version (p5'). To this end we modify again  $\mathcal{P}''_N$  as in the proof of Lemma 2.7:

$$\mathcal{P}_N := \mathcal{P}''_N - \mathfrak{Q}_N \quad (2.39)$$

for

$$\mathfrak{Q}_N = \sum_{\mathbf{r} \in \mathbb{N}_0^N} \frac{1}{\mathbf{r}!} \delta^{(\mathbf{r})}(\mathbf{x}) Q_{N,\mathbf{r}}, \quad Q_{N,\mathbf{r}} : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathbb{R}$$

such that

$$Q_{N,\mathbf{r}} u = 0 \quad \text{if } |\mathbf{r}| > \text{sc.d. } u - N, \quad (2.40)$$

$$Q_{N,\mathbf{r}+\mathbf{e}_\xi}[u] = -Q_{N,\mathbf{r}}[x^\xi u] \quad (2.41)$$

the first of which ensures that  $\mathcal{P}_N$  preserve the filtrations and the second, that  $\mathcal{P}_N$  commute with  $x^\xi$  (cf. Eqs. (2.34) and (2.35)). Solving Eq. (2.41) we obtain

$$Q_{N,\mathbf{r}} = (-1)^{|\mathbf{r}|} Q_{N,0} \circ \mathbf{x}^\mathbf{r}.$$

i.e.,  $\mathfrak{Q}_N$  is determined just by one linear functional  $Q_{N,0} : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathbb{R}$  (this fact will play also a crucial role for the reduction of our cohomological analysis in Sect. 3 to ordinary de Rham cohomologies).

Now we shall define  $Q'_{N,0}$ , inductively in  $N = 1, 2, \dots$ , so that  $\mathcal{P}_N$  satisfy (p5').

**Lemma 2.8.** *It is always possible to fulfill the condition (p5) (without prime!) by a suitable correction  $\mathfrak{Q}_{M+N}$  to  $\mathcal{P}''_{M+N}$ , which is determined by a linear functional  $Q_{M+N}$ . Furthermore, the restriction  $Q_{M+N,\mathbb{R}^M}$  of  $Q_{M+N}$  on the subspace:*

$$\mathcal{V} (\equiv \mathcal{V}_{M+N}(\mathbb{R}^M)) := \mathcal{D}'_{\mathbb{R}^M,0} \otimes \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \subset \mathcal{D}'_{temp}(\mathbb{R}^{M+N} \setminus \{\mathbf{0}\})$$

( $\mathcal{D}'_{\mathbb{R}^M,0}$  stands for the space of distributions on  $\mathbb{R}^M$  supported at zero), is uniquely determined.

*Proof.* We first define a linear map

$$\begin{aligned} \mathfrak{Q}_{M+N,\mathbb{R}^M} : \mathcal{V} &\rightarrow \mathcal{D}'_{\mathbb{R}^{M+N},0}, \\ \mathfrak{Q}_{M+N,\mathbb{R}^M}(w \otimes u) &:= \mathcal{P}''_{N+M}(w \otimes u) - w \otimes \mathcal{P}''_N u \end{aligned}$$

for  $w \in \mathcal{D}'_{\mathbb{R}^M,0}$ ,  $u \in \mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ . Thus,  $\mathfrak{Q}_{M+N,\mathbb{R}^M}$  is uniquely determined and commutes with the multiplication by the coordinates  $x^\xi$  ( $\xi = 1, \dots, M+N$ ). Hence,  $\mathfrak{Q}_{M+N,\mathbb{R}^M}$  is determined, as above, by a linear functional  $Q_{M+N,\mathbb{R}^M}$  on  $\mathcal{V}$  and such a functional is unique. To construct the full map  $\mathfrak{Q}_{N+M}$  we just need to extend the linear functional  $Q_{M+N,\mathbb{R}^M}$  from  $\mathcal{V}$  to  $\mathcal{D}'_{temp}(\mathbb{R}^{M+N} \setminus \{\mathbf{0}\})$  in such a way that the grading condition (2.40) is satisfied. This is a simple linear algebra problem and it is always possible.  $\square$

To complete the proof of the possibility to fulfill  $(p5')$  (by induction in  $N = 1, 2, \dots$ ) we note first that for  $N = 1$  the condition is trivial. Then we assume that  $(p5')$  is satisfied for  $1, \dots, N-1$ . By Lemma (2.8) for every orthogonal decomposition  $\mathbb{R}^N = \mathbf{V} + \mathbf{V}'$  we have a uniquely defined linear functional

$$Q_{N,\mathbf{V}} : \mathcal{V}_N(\mathbf{V}) \rightarrow \mathbb{R}, \quad \mathcal{V}_N(\mathbf{V}) := \mathcal{D}'_{\mathbf{V},0} \otimes \mathcal{D}'(\mathbf{V}' \setminus \{\mathbf{0}\}),$$

where  $\mathcal{D}'_{\mathbf{V},0}$  stands for the space of distributions on  $\mathbf{V}$  supported at zero. Since  $\mathcal{V}_{\mathbf{V}_1 \cap \mathbf{V}_2} = \mathcal{V}_{\mathbf{V}_1} \cap \mathcal{V}_{\mathbf{V}_2}$  and on the other hand, by the uniqueness of  $Q_{N,\mathbf{V}}$  we have:  $Q_{N,\mathbf{V}_1 \cap \mathbf{V}_2} = Q_{N,\mathbf{V}_1}|_{\mathbf{V}_1 \cap \mathbf{V}_2}$ , then the collection of linear functionals  $\{Q_{N,\mathbf{V}}\}_{\mathbf{V} \subseteq \mathbb{R}^N}$  is consistent. Hence, there exists (possibly nonunique) extension of all these functionals to a single linear functional  $Q_N : \mathcal{D}'_{temp}(\mathbb{R}^N \setminus \{\mathbf{0}\}) \rightarrow \mathbb{R}$  so that  $Q_N|_{\mathcal{V}_N(\mathbf{V})} = Q_{N,\mathbf{V}}$ . Using this functional to correct  $\mathcal{P}''_N$  as above we shall fulfill the condition  $(p5')$  by the construction.

This completes the proof of the existence of primary renormalization maps.

**Remark 2.3.** If we impose the stronger conditions of Remarks 2.1 and 2.2 then we can use a similar construction for  $\mathcal{P}_N$  as above: we should introduce the subspaces  $\mathcal{W}_{\mathbf{V}} := (\Pi_{\mathbf{V}})^* \mathcal{D}'(\mathbf{V} \setminus \{\mathbf{0}\})$  of  $\mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ , where  $\Pi_{\mathbf{V}} : \mathbb{R}^N \rightarrow \mathbf{V}$  is the orthogonal projection on  $\mathbf{V} \subset \mathbb{R}^N$ , and extend the recursively defined linear maps  $\mathcal{P}_M$  ( $1 \leq M < N$ ) consistently from every  $\mathcal{W}_{\mathbf{V}}$  to the whole space  $\mathcal{D}'(\mathbb{R}^N \setminus \{\mathbf{0}\})$ .

## 2.6 Change of renormalization maps

In this section we give a formula for change of renormalization maps. This formula is the main tool for the proof of the “universal renormalization theorem” that provides the way in which the Green functions of an arbitrary perturbative QFT change under the change of renormalization. We shall not consider the latter theorem in the present paper but in a future work.

**Theorem 2.9.** *Let  $\{P_n\}_{n=2}^\infty$  and  $\{P'_n\}_{n=2}^\infty$  be two systems of primary renormalization maps (2.23), which define the systems  $\{R_n\}_{n=2}^\infty$  and  $\{R'_n\}_{n=2}^\infty$  of renormalization maps, respectively. Then for every finite  $S \subset \mathbb{N}$  and  $G_S \in \mathcal{O}_S$  of the form (2.14) we have:*

$$R'_S G_S = \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{-partition}}} \left( R_{S/\mathfrak{P}} \otimes \text{id}_{\mathcal{D}'_{\mathfrak{P},0}} \right) \circ n.f.\mathfrak{P} \left( G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} u_{S'} \right), \quad (2.42)$$



where

$$u_{S'} = \begin{cases} Q_{S'} G_{S'} & \text{if } |S'| > 1 \\ 1 & \text{if } |S'| = 1 \end{cases}$$

and  $Q_S = (P'_S - P_S) \circ \dot{R}'_S : \mathcal{O}_S \rightarrow \mathcal{D}'_{S,0}$  for every finite  $S$  with  $|S| > 1$ .

*Explanation of the notations:* Recall that  $\mathcal{D}'_{S,0}$  stands for the space of distributions on  $\mathbf{E}_S$  supported at zero. The linear map  $Q_S$  takes values in  $\mathcal{D}'_{S,0}$  due to the difference  $P'_S - P_S$  of the primary renormalization maps and the property (p1) of their definition. For an  $S$ -partition  $\mathfrak{P}$  we have set  $\mathcal{D}'_{\mathfrak{P},0} := \bigotimes_{S' \in \mathfrak{P}} \mathcal{D}'_{S',0}$ . Thus,  $\mathcal{D}'_{\mathfrak{P},0}$  is the space of distributions over  $\mathbf{E}_S$  with

support on the partial diagonal  $\Delta_{\mathfrak{P}} := \{[x_j]_{j \in S} \in \mathbf{E}_S : x_j = x_k \text{ if } j \sim_{\mathfrak{P}} k\}$ . Then the product  $G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} u_{S'}$  takes values in the space  $\mathcal{O}_{S/\mathfrak{P}} \otimes \mathcal{D}'_{\mathfrak{P},0}$  where

we have introduced the “quotient”  $S/\mathfrak{P} := \{\min S' : S' \in \mathfrak{P}\} \subseteq S$ . More precisely, the transformation of  $G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} u_{S'}$  to the space  $\mathcal{O}_{S/\mathfrak{P}} \otimes \mathcal{D}'_{\mathfrak{P},0}$  includes

a restriction of  $G_{\mathfrak{P}}$  to the partial diagonal  $\Delta_{\mathfrak{P}}$ , with possible “transverse” derivatives due to the possible derivatives of the delta functions contained in  $u_{S'}$  ( $S' \in \mathfrak{P}$ ). The latter operation is denoted in Eq. (2.42) by  $n.f.\mathfrak{P}$  (“normal form”):

$$n.f.\mathfrak{P} : \mathcal{O}_{\mathfrak{P}} \otimes \mathcal{D}'_{\mathfrak{P},0} \rightarrow \mathcal{O}_{S/\mathfrak{P}} \otimes \mathcal{D}'_{\mathfrak{P},0}.$$

We also remind the convention  $R_1 = R_{\{k\}} = \text{id}_{\kappa} : \kappa \rightarrow \kappa$  (the identity map of the ground field  $\kappa$ ) and so, the extreme case in the sum in Eq. (2.42) when  $\mathfrak{P} = \{\{j\} : j \in S\}$  (i.e.,  $|\mathfrak{P}| = |S|$ ), corresponds to the term:  $Q_S G_S$ . Note that the term in Eq. (2.42) corresponding to  $\mathfrak{P} = \{S\}$  (i.e.,  $|\mathfrak{P}| = 1$ ) is  $R_S G_S$ .

*Proof of Theorem 2.9.* We use an induction in  $n = |S| = 2, 3, \dots$ . For  $n = 2$  Eq. (2.42) reduces to the equation  $R'_2 = R_2 + Q_2$  and  $Q_2 = P'_2 - P_2$ . But  $R_2 = P_2$  and  $R'_2 = P'_2$ , by the construction of renormalization maps, and  $\dot{R}'_2$  is just the inclusion  $\mathcal{O}_2 \hookrightarrow \mathcal{D}'(\mathbf{E}_2 \setminus \{\mathbf{0}\})$ .

Now let  $n > 2$  and assume Eq. (2.42) is proven for all finite subsets  $S' \subset \mathbb{N}$  with  $|S'| < n$ . We shall first prove the equality:

$$\dot{R}'_S G_S = \sum_{\substack{\mathfrak{P} \text{ is a proper} \\ S\text{-partition}}} \left( \dot{R}_{S/\mathfrak{P}} \otimes \text{id}_{\mathcal{D}'_{\mathfrak{P},0}} \right) \circ n.f.\mathfrak{P} \left( G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} u_{S'} \right). \quad (2.43)$$

Note that Eq. (2.42) implies (2.43) because of the fact that  $\dot{R}'_S G_S = (R_S$

$G_S) \big|_{\mathbf{E}_S \setminus \{0\}}$ . Due to the covering property (2.18) it is enough to prove that

$$\text{left hand side of Eq. (2.43)} \big|_{\mathbf{F}_{\mathfrak{P}}} = \text{right hand side of Eq. (2.43)} \big|_{\mathbf{F}_{\mathfrak{P}}} \quad (2.44)$$

for every proper  $S$ -partition  $\mathfrak{P}$ .

For the restriction of the right hand side we obtain

$$\sum_{\mathfrak{P}' \leq \mathfrak{P}} \left[ \left( \dot{R}_{S/\mathfrak{P}'} \otimes \text{id}_{\mathcal{D}'_{\mathfrak{P}',0}} \right) \circ n.f.\mathfrak{P}' \left( G_{\mathfrak{P}'} \prod_{S' \in \mathfrak{P}'} u_{S'} \right) \right] \big|_{\mathbf{F}_{\mathfrak{P}}}, \quad (2.45)$$

where the relation  $\mathfrak{P}' \leq \mathfrak{P}$  for two  $S$ -partitions stands for:

$$\mathfrak{P}' \leq \mathfrak{P} \quad \stackrel{\text{def}}{\iff} \quad j \sim_{\mathfrak{P}'} k \text{ implies } j \sim_{\mathfrak{P}} k \quad (\forall j, k \in S). \quad (2.46)$$

This is because the support of  $\prod_{S' \in \mathfrak{P}'} u_{S'}$  for  $\mathfrak{P}' \not\leq \mathfrak{P}$  is disjoint from  $\mathbf{F}_{\mathfrak{P}}$ .

The restriction of the left hand side of Eq. (2.42) is computed accordingly to property (r4) of renormalization maps (Sect. 2.3). We thus obtain that for every proper  $S$ -partition  $\mathfrak{P}$  we have:

$$\dot{R}'_S G_S \big|_{\mathbf{F}_{\mathfrak{P}}} = R'_S G_S \big|_{\mathbf{F}_{\mathfrak{P}}} = \left( G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} R'_{S'} G_{S'} \right) \big|_{\mathbf{F}_{\mathfrak{P}}}.$$

By the inductive assumption we get that  $\left( G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} R'_{S'} G_{S'} \right) \big|_{\mathbf{F}_{\mathfrak{P}}}$  equals

$$\begin{aligned} & G_{\mathfrak{P}} \cdot \left[ \prod_{S' \in \mathfrak{P}} \sum_{\substack{\mathfrak{P}_{S'} \text{ is a} \\ S'\text{-partition}}} \left( R_{S'/\mathfrak{P}_{S'}} \otimes \text{id}_{\mathcal{D}'_{\mathfrak{P}_{S'},0}} \right) \right. \\ & \quad \left. \circ n.f.\mathfrak{P}_{S'} \left( G_{\mathfrak{P}_{S'}} \prod_{S'' \in \mathfrak{P}_{S'}} u_{S''} \right) \right] \big|_{\mathbf{F}_{\mathfrak{P}}}. \end{aligned}$$

Now we expand the product in  $S' \in \mathfrak{P}$  of the sums and combine all the  $S'$ -partitions  $\mathfrak{P}_{S'}$  into a single  $S$ -partition  $\mathfrak{P}' := \bigcup_{S' \in \mathfrak{P}} \mathfrak{P}_{S'}$ . Taking also

into account that  $G_{\mathfrak{P}} \cdot \prod_{S' \in \mathfrak{P}} G_{\mathfrak{P}_{S'}} = G_{\mathfrak{P}'}$  together with the property (r4) we arrive at the expression (2.45).

Having proven Eq. (2.43) for  $|S| = n$  we can prove Eq. (2.42) by using the compositions  $R_S = P_S \circ \dot{R}_S$  and  $R'_S = P'_S \circ \dot{R}'_S$ . This is done as follows:

$$R'_S G_S = Q_S G_S + P_S \dot{R}'_S G_S$$

$$\begin{aligned}
 &= Q_S G_S + P_S \sum_{\substack{\mathfrak{P} \text{ is a proper} \\ S\text{-partition}}} \left( \dot{R}_{S/\mathfrak{P}} \otimes \text{id}_{\mathcal{D}'_{\mathfrak{P},0}} \right) \circ n.f.\mathfrak{P} \left( G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} u_{S'} \right) \\
 &= Q_S G_S + \sum_{\substack{\mathfrak{P} \text{ is a proper} \\ S\text{-partition}}} \left( (P_{S/\mathfrak{P}} \circ \dot{R}_{S/\mathfrak{P}}) \otimes \text{id}_{\mathcal{D}'_{\mathfrak{P},0}} \right) \circ n.f.\mathfrak{P} \left( G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} u_{S'} \right),
 \end{aligned}$$

where in the last step we have used the property (p5) of primary renormalization maps (Sect. 2.4). Thus, we arrive at the expression of the right hand side of Eq. (2.42). This completes the proof of Theorem 2.9.  $\square$

We see that a change of renormalization maps  $\{R_S\} \rightarrow \{R'_S\}$  is completely determined by a system of linear maps

$$Q_S : \mathcal{O}_S \rightarrow \mathcal{D}'_{S,0} \quad (2.47)$$

indexed by nonempty finite subsets  $S$  of  $\mathbb{N}$ . They are related to  $\{R_S\}$  and  $\{R'_S\}$  by

$$Q_S := (P'_S - P_S) \circ \dot{R}'_S \quad \text{for } |S| > 1 \quad \text{and} \quad Q_S := 1 \quad \text{for } |S| = 1.$$

Let us point out that the primary renormalization maps contain more information than those, which is encoded in the renormalization maps.

The conditions that characterize the system  $\{Q_S\}$  (2.47) are:

(c1) Permutation symmetry: for every bijection  $\sigma : S \cong S'$  we have

$$\sigma^* \circ Q_{S'} = Q_S \circ \sigma^*$$

$$((\sigma^* F)(x_{j_1}, \dots, x_{j_n}) := F(x_{\sigma(j_1)}, \dots, x_{\sigma(j_n)})).$$

So, by (c1) all  $Q_S$  are characterized by  $Q_n := Q_{\{1, \dots, n\}}$  for  $n = |S| = 2, 3, \dots$

(c2) Preservation of the filtrations:  $Q_S \mathcal{F}_\ell \mathcal{O}_S \subseteq \mathcal{F}_\ell \mathcal{D}'_S$ .

(c3) For every polynomial  $f \in \kappa[\mathbf{E}_n]$  and  $G \in \mathcal{O}_n$  we have:

$$Q_n f G = f Q_n G.$$

There is a converse statement:

**Proposition 2.10.** *For every system  $\{Q_S\}$  satisfying the above conditions (c1)–(c3) and a given system of renormalization maps  $\{R_S\}$  there exists a system of primary renormalization maps  $\{P'_S\}$ , which determines a system*

of renormalization maps  $\{R'_S\}$  so that the maps  $\{Q_S\}$  correspond to the change  $\{R_S\} \rightarrow \{R'_S\}$ .

Since we shall not use here the above proposition we shall only sketch its *proof* (there is also a similarity with the proof of Theorem 3.1 in the next section). We construct the primary renormalization maps  $P_n$  by induction in  $n = 2, 3, \dots$ . For  $n = 2$ ,  $\dot{R}_2$  is the embedding  $\mathcal{O}_2 \hookrightarrow \mathcal{D}'(\mathbf{E}_2 \setminus \{\mathbf{0}\})$  and hence,  $P'_2 = P_2 + Q_2 \circ (\dot{R}_2)^{-1}$ . Having constructed  $P'_m$  for  $m = 2, \dots, n-1$  we define  $P'_n$  as  $P_n + \tilde{Q}_n$ , where the linear map  $\tilde{Q}_n$  is an extension of  $Q_n \circ (\dot{R}'_n)^{-1}$  from the subspace  $\dot{R}'_S(\mathcal{O}_n) \subset \mathcal{D}'(\mathbf{E}_N \setminus \{\mathbf{0}\})$  to the whole space. The later extension is not arbitrary but is done in the way used in the proof of Theorem 3.1 for the construction of the analogous map  $\tilde{Q}_n$  there. This completes the proof.

One can further relate to the systems  $\{Q_S\}$  a certain group product: if  $\{Q'_S\}$  characterizes another change of renormalization maps,  $\{R'_S\} \rightarrow \{R''_S\}$ , then a question arises what is  $\{Q''_S\}$  characterizing the change  $\{R_S\} \rightarrow \{R''_S\}$ ? The answer is analogous to Eq. (2.42):

$$Q''_S G_S = \sum_{\substack{\mathfrak{P} \text{ is a} \\ S\text{-partition}}} \left( Q'_{S/\mathfrak{P}} \otimes \text{id}_{\mathcal{D}'_{\mathfrak{P},0}} \right) \circ n.f.\mathfrak{P} \left( G_{\mathfrak{P}} \prod_{S' \in \mathfrak{P}} Q_{S'} G_{S'} \right). \quad (2.48)$$

The method for proving the composition law (2.48) is the same as those, which is used in the proof of Theorem 2.9. It is natural to expect that the set of all systems of linear maps  $\{Q_S\}$  satisfying the conditions (c1)–(c3) form a group under the above law with a unit  $\{Q_1 = 1, Q_n = 0 \text{ for } n > 1\}$ . This group can be called **universal renormalization group** and we intend to study it in a separate work. (See also Remark 3.2 at the end of Sect. 3.3.)

## 2.7 Remark on renormalization on Riemann manifolds

Our renormalization scheme based on the Epstein–Glaser approach can be generalized for renormalization on Riemann manifolds. We shall briefly sketch this construction here. The restriction of translation invariance we have used up to now is not crucial but only simplifies the considerations.

If  $\mathcal{M}$  is a Riemann manifold we should introduce the algebra  $\mathcal{O} \equiv \mathcal{O}_2$  as a subalgebra of  $\mathcal{C}^\infty(F_2(\mathcal{M}))$ . We should also assume that  $\mathcal{O}$  is invariant under the actions of all smooth vector fields on  $\mathcal{M} \times \mathcal{M}$ . Then the algebras  $\mathcal{O}_n$  and  $\mathcal{O}_S$  for finite  $S \subset \mathbb{N}$  are constructed in the same way as for the flat case:  $\mathcal{O}_n$  is the subalgebra of  $\mathcal{C}^\infty(F_n(\mathcal{M}))$  generated by all embeddings  $\hat{\iota}_{jk} \mathcal{O}_2$  for  $1 \leq j < k \leq n$ .

The scaling degree for distributions on  $\mathcal{M}^{\times n}$  is introduced with respect to the total diagonal

$$\Delta_n(\mathcal{M}) := \{(x, \dots, x) : x \in \mathcal{M}\} \subset \mathcal{M}^{\times n}.$$

Then the filtrations on the distributions spaces and on the algebras  $\mathcal{O}_n$  are introduced similarly to the flat case.

Concerning the preliminary definition of renormalization maps, conditions (r1), (r2) and (r4) remains the same (the sets  $\mathbf{F}_{\mathfrak{P}}$  are defined now as an open covering of  $\mathcal{M}^{\times n} \setminus \Delta_n(\mathcal{M})$ ). Condition (r3) should be extended not only for polynomials, but for arbitrary smooth functions on  $\mathcal{M} \times \mathcal{M}$ , i.e.:

$$R_n f(x_j, x_k) G(x_1, \dots, x_n) = f(x_j, x_k) R_n G(x_1, \dots, x_n)$$

for every  $f \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$ ,  $G \in \mathcal{O}_n$  and  $1 \leq j < k \leq n$ .

Then again we have linear maps

$$\dot{R}_n : \mathcal{O}_n \rightarrow \mathcal{D}'_{temp}(\mathcal{M}^{\times n} \setminus \Delta_n(\mathcal{M}))$$

determined by condition (r4) from  $R_2, \dots, R_{n-1}$ . Thus, the primary renormalization maps we need to construct inductively  $R_n$  are linear maps:

$$P_n : \mathcal{D}'_{temp}(\mathcal{M}^{\times n} \setminus \Delta_n(\mathcal{M})) \rightarrow \mathcal{D}'(\mathcal{M}^{\times n}).$$

These maps should satisfy the following properties:

- $\mathcal{S}_n$ -symmetry.
- Preservation of filtrations.
- For every  $f \in \mathcal{C}^\infty(\mathcal{M}^{\times n})$  and  $u \in \mathcal{D}'_{temp}(\mathcal{M}^{\times n} \setminus \Delta_n(\mathcal{M}))$ :

$$P_n f u = f P_n u.$$

- Let  $\mathfrak{P}$  be an  $n$ -partition and set  $S := \{\min S' : S' \in \mathfrak{P}\}$ . Then

$$P_n \left[ \left( \bigotimes_{S' \in \mathfrak{P}} u_{S'} \right) \otimes u \right] = \left( \bigotimes_{S' \in \mathfrak{P}} u_{S'} \right) \otimes P_S u,$$

where  $u \in \mathcal{D}'_{temp}(\mathcal{M}^S \setminus \Delta_S(\mathcal{M}))$  and  $u_{S'} \in \mathcal{D}'(\mathcal{M}^{S'})$  with  $\text{supp } u_{S'} \subseteq \Delta_{S'}(\mathcal{M})$  (the total diagonal in the cartesian power  $\mathcal{M}^{S'}$ ) for every  $S' \in \mathfrak{P}$ .

We then set

$$R_2 = P_2, \quad R_n = P_n \circ \dot{R}_n \quad (n > 2).$$

Apart from the above conditions on  $R_n$  and  $P_n$  it is also natural to impose in the curved case the so called “*general covariance*”. This assumes that we have the linear maps  $R_n =: R_{n(\mathcal{M})}$  and  $P_n =: P_{n(\mathcal{M})}$  defined for every Riemann manifold  $\mathcal{M}$  and furthermore, they depend *naturally* on  $\mathcal{M}$  under isometric embeddings  $\mathcal{M}' \hookrightarrow \mathcal{M}$ .

We shall not consider further here the problem of constructing such families of renormalization maps but we believe that this can be done using the methods of this work combined with the techniques used in the causal perturbative QFT on pseudo-Riemann manifolds (for instance, using Riemann normal coordinates).

### 3 Anomalies in QFT and cohomologies of configuration spaces

When a symmetry of unrenormalized (bare) Feynman amplitudes is broken after the renormalization one speaks about an *anomaly* in the theory. Clearly, the source of the anomalies in perturbative QFT is the absence of commutativity between the action of the partial differential operators and the renormalization, i.e., the “commutators”

$$c_n[A] = A \circ R_n - R_n \circ A,$$

are generally nonzero for linear partial differential operators  $A$  on  $\mathbf{E}_n$ . Note that by the extension property  $R_n G|_{\mathbf{F}_n} = G$  (cf. Sect. 2.3) for  $G \in \mathcal{O}_n$  it follows that  $c_n[A]$  is a linear map

$$c_n[A] : \mathcal{O}_n \rightarrow \mathcal{D}'[\widehat{\Delta}_n], \quad \widehat{\Delta}_n := \mathbf{E}_n \setminus \mathbf{F}_n$$

( $\widehat{\Delta}_n$  is the so called “large” diagonal in  $\mathbf{E}_n$ ), where  $\mathcal{D}'[\widehat{\Delta}_n]$  stands for the space of distributions on  $\mathbf{E}_n$  supported at  $\widehat{\Delta}_n$ . It is straightforward to observe that  $c_n[A]$  is a Hochschild 1-cocycle for the associative algebra of all linear partial differential operators on  $\mathbf{E}_n$  (having polynomial coefficients):

$$A_1 \circ c_n[A_2] - c_n[A_1 \circ A_2] + c_n[A_1] \circ A_2 = 0 \quad (3.1)$$

(hence,  $c_n[A]$  is a Hochschild 1-cocycle in the bimodule of all linear maps  $\mathcal{O}_n \rightarrow \mathcal{D}'[\widehat{\Delta}_n]$ ). It is also clear that every change of the renormalization map  $R_n \rightarrow R'_n$  changes the cocycle  $c_n$  by a coboundary:

$$c_n[A] - c'_n[A] = A \circ b - b \circ A,$$

where  $b = R_n - R'_n$ . Thus, we shall call the maps  $c_n$  linearly depending on partial differential operators *renormalization cocycles*.

Nevertheless, it is not so simple to take the full renormalization ambiguity into account: as we mentioned in the previous section, condition (r4) is nonlinear and hence, the class

$$\{\{c_n\} : \{c_n\} \text{ is generated by } \{R_n\}\}$$

forms a nonlinear subset in the direct sum of the cohomology classes of all  $c_n$ . In this section we shall find a description of the above class of renormalization cocycles.

### 3.1 Cohomological equations

By property (r3) (Sect. 2.3) of the renormalization maps we have  $c_n[x_k^\mu] = 0$ , where  $x_k^\mu$  for  $k = 1, \dots, n-1$ ,  $\mu = 1, \dots, D$  are the coordinates in  $\mathbf{E}_n \cong \mathbb{R}^{D(n-1)}$  according to the isomorphism (2.2). Thus, what remains to be determined (due to Eq. (3.1)) is

$$\omega_{n;k,\mu} := c_n[\partial_{x_k^\mu}] = [\partial_{x_k^\mu}, R_n] \quad (3.2)$$

( $\partial_{x_k^\mu} := \frac{\partial}{\partial x_k^\mu}$ ). For short, we denote the pair of indices  $(k, \mu)$  in Eq. (3.2) by a single index  $\xi$  (or  $\eta, \dots$ ) running from 1 to  $D(n-1)$ ; then the corresponding components  $x_k^\mu$  will be denoted by  $x^\xi$ . In what follows we shall call the above system of linear maps  $\omega_{n;\xi}$  **renormalization cocycles**.

Applying to the definition of  $\omega_{n;\xi}$  the construction of renormalization maps by Theorem 2.3,  $R_n = P_n \circ \dot{R}_n$  ( $n > 2$ ), we obtain a decomposition:

$$\omega_{n;\xi} = \gamma_{n;\xi} + \dot{\omega}_{n;\xi} \quad (n > 2), \quad \omega_{2;\xi} \equiv \gamma_{2;\xi}, \quad (3.3)$$

$$\gamma_{n;\xi} := [\partial_{x^\xi}, P_n] \circ \dot{R}_n \quad (n > 2), \quad (3.4)$$

$$\dot{\omega}_{n;\xi} := P_n \circ [\partial_{x^\xi}, \dot{R}_n] \quad (n > 2).$$

The linear maps  $\gamma_{n;\xi}$  are simpler than  $\omega_{n;\xi}$  since they take values that are distributions supported at the *reduced total diagonal*, i.e., at the origin  $\mathbf{0} \in \mathbf{E}_n$  (this is due to condition (p1)):

$$\gamma_{n;\xi} : \mathcal{O}_n \rightarrow \mathcal{D}'_{n,0} \quad (3.5)$$

(recall that  $\mathcal{D}'_{n,0}$  stands for the space distributions on  $\mathbf{E}_n$  supported at zero). On the other hand, the remaining part  $\dot{\omega}_{n;\xi}$  of  $\omega_{n;\xi}$  in Eq. (3.3) is determined

by the renormalization induction. This is first because of the presence of  $\dot{R}_n$  and second, due to the commutator  $[\partial_{x^\xi}, \dot{R}_n]$ , which produces at least one delta function (with possible derivatives) and then by property (p5)  $P_n$  is reduced to  $P_{n'}$  with  $n' < n$ . So, the only new information is contained in  $\gamma_{n;\xi}$ .

The linear maps  $\gamma_{n;\xi}$  satisfy the “differential equations”:

$$\begin{aligned} & [\partial_{x^\xi}, \gamma_{2;\eta}] - [\partial_{x^\eta}, \gamma_{2;\xi}] = 0, \\ & [\partial_{x^\xi}, \gamma_{n;\eta}] - [\partial_{x^\eta}, \gamma_{n;\xi}] \\ & = -[\partial_{x^\xi}, P_n] \circ [\partial_{x^\eta}, \dot{R}_n] + [\partial_{x^\eta}, P_n] \circ [\partial_{x^\xi}, \dot{R}_n] \quad (n > 2) \end{aligned} \tag{3.6}$$

for all  $\xi, \eta = 1, \dots, D(n-1)$ , which are derived by a straightforward computation. We shall characterize  $\gamma_{n;\xi}$  by these equations. Before that, let us point out that *the right hand side of (3.6) is determined by the renormalization induction*. The reason is the same as above: the values of  $[\partial_{x^\xi}, \dot{R}_n]$  are distributions supported at the large diagonal  $\hat{\Delta}_n$  and then,  $[\partial_{x^\xi}, P_n]$  act on functions whose nontrivial dependence is in less than  $n-1$  relative distances (i.e., functions on  $\mathbf{E}_S$  with  $|S| < n$ ). Thus, we can consider (3.6) as equations for  $\{\gamma_{n;\xi}\}_\xi$  for fixed  $n$ , whose right hand side is determined by  $\{\gamma_{n';\xi}\}_\xi$  for  $n = 2, \dots, n-1$ . We shall call the maps  $\gamma_{n;\xi}$  **primary renormalization cocycles** without meaning of closedness with respect to some differential.

There are important additional restrictions to the solutions of Eqs. (3.6), which are important for us. These are the conditions

$$[x^\eta, \gamma_{n;\xi}] = 0, \tag{3.7}$$

$$\gamma_{n;\xi} \mathcal{F}_\ell \mathcal{O}_n \subseteq \mathcal{F}_{\ell+1} \mathcal{D}'_{n,0} \tag{3.8}$$

( $\xi, \eta = 1, \dots, D(n-1)$ ,  $\ell = 0, 1, \dots$ ), which have to satisfy all renormalization cocycles defined by (3.4) (the first one is due to Eqs. (3.5) and conditions (r3) and (p4), and the second one is due to (r2), (p2) and Eq. (2.7)).

**Theorem 3.1.** *Let  $n > 2$  and we have a system of primary renormalization maps  $P_2, P_3, \dots, P_n$  (which therefore determine renormalization maps  $R_2, R_3, \dots, R_n$ ). Let  $\{\gamma_{n;\xi}\}_\xi$  be defined accordingly to Eq. (3.4) and  $\{\gamma'_{n;\xi}\}_\xi$  be a solution of Eqs. (3.6), (3.7) and (3.8), which differs from  $\{\gamma_{n;\xi}\}_\xi$  by an exact solution, i.e., the difference  $\gamma'_{n;\xi} - \gamma_{n;\xi}$  is of a form*

$$\gamma'_{n;\xi} - \gamma_{n;\xi} = [\partial_{x^\xi}, Q_n] \tag{3.9}$$

( $\xi = 1, \dots, D(n-1)$ ), for some linear map

$$Q_n : \mathcal{O}_n \rightarrow \mathcal{D}'_{n,0}.$$



Then there exists a primary renormalization map  $P'_n$ , which together with  $P_2, \dots, P_{n-1}$  determines a system of renormalization maps  $R_2, \dots, R_{n-1}$  and  $R'_n$  and a primary renormalization cocycle coinciding with  $\{\gamma'_{n;\xi}\}_\xi$ .

*Proof.* Equations (3.6) have an obvious form of cohomological equations. If  $\{\gamma_{n;\xi}\}_\xi$  is a solution of them for some fixed  $n$  then  $\{\gamma'_{n;\xi}\}_\xi$  related to  $\{\gamma_{n;\xi}\}_\xi$  by Eq. (3.9) is a solution too. If  $\{\gamma_{n;\xi}\}_\xi$  satisfies conditions (3.7) and (3.8) and  $Q_n$  satisfy also

$$[x^\eta, Q_n] = 0, \quad Q_n \mathcal{F}_\ell \mathcal{O}_n \subseteq \mathcal{F}_\ell \mathcal{D}'_{\mathbb{R}^N, 0}$$

then  $\{\gamma'_{n;\xi}\}_\xi$  also obey Eqs. (3.7) and (3.8).

We shall show now that if the solution  $\{\gamma_{n;\xi}\}_\xi$  corresponds by Eq. (3.4) to some renormalization map  $R_n$  then the changed solution  $\{\gamma'_{n;\xi}\}_\xi$  (3.9) corresponds to another renormalization map  $R'_n : \mathcal{O}_n \rightarrow \mathcal{D}'_n$ .

To this end let us introduce the subspace  $\mathfrak{S} \subset \mathcal{D}'(\mathbf{E}_n \setminus \{\mathbf{0}\})$  of distributions that are supported at unions of linear subspaces of  $\mathbf{E}_n$ . Thus,  $\mathcal{D}'[\hat{\Delta}_n \setminus \{\mathbf{0}\}] \subset \mathfrak{S}$ . We observe next that because  $\dot{R}_n$  is an injection  $\mathcal{O}_n \hookrightarrow \mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\})$  then there exists a linear map

$$\tilde{Q}_n : \mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\}) \rightarrow \mathcal{D}'_{n,0} \quad (3.10)$$

such that

$$Q_n = \tilde{Q}_n \circ \dot{R}_n.$$

Moreover, since  $\mathfrak{S} \cap \dot{R}_n(\mathcal{O}_n) = \{0\}$  we can additionally choose  $\tilde{Q}_n$  in such a way that

$$\tilde{Q}_n|_{\mathfrak{S}} = 0. \quad (3.11)$$

In fact,  $\tilde{Q}_n$  can be constructed by a linear map

$$\mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\}) / \mathfrak{S} \rightarrow \mathcal{D}'_{n,0} \quad (3.12)$$

that extends the map  $Q_n$  under the embedding

$$\mathcal{O}_n \hookrightarrow \mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\}) / \mathfrak{S}. \quad (3.13)$$

We claim also that it is possible to construct  $\tilde{Q}_n$  also in such a way that it additionally satisfies the following conditions: (a)  $\tilde{Q}_n$  preserves the gradings,  $\tilde{Q}_n \mathcal{F}_\ell \mathcal{D}'_{temp}(\mathbf{E}_n \setminus \{\mathbf{0}\}) \subseteq \mathcal{F}_\ell \mathcal{D}'_{n,0}$  ( $\ell = 0, 1, \dots$ ); (b)  $\tilde{Q}_n$  commutes with all  $x^\eta$ ,  $[x^\eta, Q_n] = 0$ ; (c)  $\tilde{Q}_n$  is invariant under orthogonal transformations of  $\mathbf{E}_n$ . Indeed, to achieve (a) we note that the linear map (3.13) preserves the

gradings and so, the extension (3.12) can be made to preserve them too. Regarding (b), we use the arguments of Lemma 2.7 in order to construct the extension (3.12) in such a way that it also commutes with  $x^\eta$ . Finally, (c) can be simply achieved by the averaging operation used in Sect. 2.5.

Thus, we can introduce a new primary renormalization map

$$P'_n = P_n + \tilde{Q}_n.$$

and it satisfies all conditions (p1)–(p5) by the construction. (For instance, the condition (p5) is ensured by condition (3.11).) Due of Eq. (3.11) we have also for  $P'_n$ :

$$[\partial_{x^\xi}, P'_n] \circ \dot{R}_n = \gamma'_{n;\xi} - \tilde{Q}_n \circ [\partial_{x^\xi}, \dot{R}_n] = \gamma'_{n;\xi},$$

since the image of  $[\partial_{x^\xi}, \dot{R}_n]$  is contained in the space  $\mathcal{D}'[\hat{\Delta}_n \setminus \{\mathbf{0}\}] \subset \mathfrak{S}$ . We obtained that  $\gamma'_{n;\xi}$  correspond by Eq. (3.4) to another renormalization map  $R'_n = P'_n \circ \dot{R}_n$ . This completes the proof of the theorem.  $\square$

Thus, we have seen that the linear maps  $\gamma_{n;\xi}$  (for fixed  $n$ ), corresponding to renormalization maps, are characterized by solutions of Eqs. (3.6) modulo “exact 1-cocycles” i.e., maps  $c_{n;\xi}$  that satisfy the equations:

$$[\partial_{x^\xi}, c_{n;\eta}] - [\partial_{x^\eta}, c_{n;\xi}] = 0$$

(for  $\xi, \eta = 1, \dots, D(n-1)$ ). In the next subsection we shall reduce the corresponding de Rham cohomologies to simpler ones.

### 3.2 De Rham cohomologies of differential modules

The linear maps  $\gamma_{n;\xi}$  (3.5) that determine the renormalization cocycles  $\omega_{n;\xi}$  take values that are distributions supported at  $\mathbf{0} \in \mathbf{E}_n$ , i.e. belonging to the space  $\mathcal{D}'_{n,0}$ . Let us expand them in delta functions and their derivatives:

$$\gamma_{n;\xi} = \sum_{\mathbf{r} \in \mathbb{N}_0^N} \frac{1}{\mathbf{r}!} \delta^{(\mathbf{r})}(\mathbf{x}) \Gamma_{n;\xi;\mathbf{r}}, \quad \Gamma_{n;\xi;\mathbf{r}} : \mathcal{O}_n \rightarrow \mathbb{C} \quad (3.14)$$

( $N = D(n-1)$ ,  $\delta^{(\mathbf{r})}(\mathbf{x}) := \partial_{\mathbf{x}}^{\mathbf{r}} \delta(\mathbf{x})$ ,  $\partial_{\mathbf{x}}^{\mathbf{r}} := \prod_{\xi} \partial_{x^\xi}^{r_\xi}$ ). Note that the sum in (3.14) becomes finite only after applying both sides to a function. Thus we get a characterization of  $\gamma_{n;\xi}$  by an infinite set of linear functionals  $\Gamma_{n;\xi;\mathbf{r}}$  on  $\mathcal{O}_n$ . In fact, we shall show now that the whole information about  $\gamma_{n;\xi}$  is contained in the leading term  $\Gamma_{n;\xi;0}$  of the expansion (3.14). This follows

if we take into account the relation  $[x^\eta, \gamma_{n;\xi}] = 0$ . Combining the latter identity with (3.14) we obtain a recursive relation  $-(r_\eta + 1) \Gamma_{n;\xi;\mathbf{r}+\mathbf{e}_\eta} = \Gamma_{n;\xi;\mathbf{r}} \circ x^\eta$ , which then implies that

$$\Gamma_{n;\xi;\mathbf{r}} = (-1)^{|\mathbf{r}|} \Gamma_{n;\xi;0} \circ \mathbf{x}^\mathbf{r} \quad (3.15)$$

( $\mathbf{x}^\mathbf{r} := \prod_\xi (x^\xi)^{r_\xi}$ ). We set

$$\Gamma_{n;\xi} := \Gamma_{n;\xi;0} \quad (3.16)$$

for  $\xi = 1, \dots, D(n-1)$  and organize them as a 1-form

$$\mathbf{\Gamma}_n := \sum_{\xi=1}^N \Gamma_{n;\xi} dx^\xi \quad (3.17)$$

with coefficients in the dual differential module  $\mathcal{O}'_n$ , i.e.,

$$\mathbf{\Gamma}_n \in \Omega^1(\mathcal{O}'_n).$$

In more details, we fix the above notions in the following definitions.

**Definition 3.1.** Let us introduce the associative  $\kappa$ -algebra  $\mathfrak{D}_N$  ( $N \in \mathbb{N}$ ) of all linear partial differential operators over  $\mathbb{R}^N$  with polynomial coefficients belonging to  $\kappa[\mathbf{x}]$  (recall that  $\kappa$  is the ground field  $\kappa \subset \mathbb{R}$ ). A  $\mathbb{Z}$ -**filtered**<sup>3</sup>  $\mathfrak{D}_N$ -module is a module  $\mathcal{N}$  of  $\mathfrak{D}_N$ , which is endowed with an increasing filtration

$$\mathcal{N} = \bigcup_{\ell \in \mathbb{Z}} \mathcal{F}_\ell \mathcal{N}, \quad \mathcal{F}_\ell \mathcal{N} \subseteq \mathcal{F}_{\ell+1} \mathcal{N},$$

such that for every  $A \in \mathfrak{D}_N$  and  $u \in \mathcal{N}$  we have

$$\deg Au \leq \deg A + \deg u, \quad (3.18)$$

where

$$\deg u := \min \{ \ell : u \in \mathcal{F}_\ell \mathcal{N} \}$$

and the scaling degree of a differential operator is defined by:

$$\deg \left( \sum_{\mathbf{r} \in \mathbb{N}_0^N} f_\mathbf{r}(\mathbf{x}) \partial^\mathbf{r} \right) = \max_{\mathbf{r} \in \mathbb{N}_0^N} \{ |\mathbf{r}| + \text{sc.d. } f_\mathbf{r} \}.$$

The algebras  $\mathcal{O}_n$  and the distribution spaces  $\mathcal{D}'$  considered in Sect. 2.2 give us examples of  $\mathbb{Z}$ -filtered  $\mathfrak{D}_N$ -modules (for  $\mathcal{O}_n$ :  $N = D(n-1)$ ). In

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<sup>3</sup>in the previous section we used  $\mathbb{R}$ -filtrations but here it will be sufficient to restrict them to  $\mathbb{Z}$

particular,  $\mathcal{D}'_{\mathbb{R}^N,0}$  becomes a  $\mathbb{Z}$ -filtered  $\mathfrak{D}_N$ -module in which  $\mathcal{F}_\ell \mathcal{D}'_{\mathbb{R}^N,0} = \{0\}$  for  $\ell < N$ .

Thus,  $\Gamma_{n;\xi}$  belong to the dual module of  $\mathcal{O}_n$  generally defined by:

**Definition 3.2.** For a  $\mathfrak{D}_N$ -module  $\mathcal{N}$  the **dual** module is the algebraic dual space  $\mathcal{N}'$  endowed with the dual action of  $\mathfrak{D}_N$ : for  $\Phi \in \mathcal{N}'$  and  $\xi = 1, \dots, N$  the dual actions of  $x^\xi$  and  $\partial_{x^\xi}$  are  $x^\xi(\Phi) := \Phi \circ x^\xi$  and  $\partial_{x^\xi}(\Phi) := -\Phi \circ \partial_{x^\xi}$ , respectively.

Let us point out that the dual differential module  $\mathcal{N}'$  of a  $\mathbb{Z}$ -filtered  $\mathfrak{D}_N$ -module  $\mathcal{N}$  is not naturally  $\mathbb{Z}$ -graded. But it has a differential submodule that is  $\mathbb{Z}$ -graded. A simple computation shows that

$$\mathcal{N}^\bullet := \bigcup_{\ell \in \mathbb{Z}} (\mathcal{F}_\ell \mathcal{N})^\perp \quad (\subseteq \mathcal{N}'), \quad (\mathcal{F}_\ell \mathcal{N})^\perp := \left\{ \Phi \in \mathcal{N}' : \Phi|_{\mathcal{F}_\ell \mathcal{N}} = 0 \right\} \quad (3.19)$$

is a  $\mathfrak{D}_N$ -submodule of  $\mathcal{N}'$  and it becomes  $\mathbb{Z}$ -filtered with an increasing  $\mathbb{Z}$ -filtration if we set

$$\mathcal{F}_\ell \mathcal{N}^\bullet := (\mathcal{F}_{-\ell} \mathcal{N})^\perp.$$

The linear functionals  $\Gamma_{n;\xi}$  that characterize the primary renormalization cocycles  $\gamma_{n;\xi}$  are elements of  $(\mathcal{O}_n)^\bullet$ . We shall show this in a slightly more general situation.

Let  $\mathcal{N}$  be a  $\mathbb{Z}$ -graded  $\mathfrak{D}_N$ -module and denote

$$\mathcal{R}_N(\mathcal{N}) := \left\{ \phi : \phi \text{ linearly maps } \mathcal{N} \text{ to } \mathcal{D}'_{\mathbb{R}^N,0}, [x^\xi, \phi] = 0 \ (\forall \xi = 1, \dots, N), \right. \\ \left. \text{and } \exists L \in \mathbb{Z} \text{ such that } \phi \mathcal{F}_\ell \mathcal{N} \subseteq \mathcal{F}_{\ell+L} \mathcal{D}'_{\mathbb{R}^N,0} \ (\forall \ell \in \mathbb{Z}) \right\}.$$

(so for example,  $\gamma_{n;\xi} \in \mathcal{R}_N(\mathcal{O}_n)$  for  $N = D(n-1)$ ). Expand  $\phi \in \mathcal{R}_N(\mathcal{N})$  in delta functions and their derivatives:

$$\phi = \sum_{\mathbf{r} \in \mathbb{N}_0^N} \frac{1}{\mathbf{r}!} \delta^{(\mathbf{r})}(\mathbf{x}) \Phi_{\mathbf{r}}, \quad \Phi_{\mathbf{r}} : \mathcal{N} \rightarrow \mathbb{R}. \quad (3.20)$$

Then the assignment

$$\phi \mapsto \Phi_0 \quad (3.21)$$

is injective and  $\phi$  is determined by  $\Phi_0$  by the formula

$$\Phi_{\mathbf{r}} = (-1)^{|\mathbf{r}|} \Phi_0 \circ \mathbf{x}^{\mathbf{r}}. \quad (3.22)$$

This is proven exactly as above for the case of  $\gamma_{n;\xi}$ . Furthermore, under the assignment (3.21):

$$\text{if } \phi \mapsto \Phi_0 \quad \text{then} \quad [\partial_{x^\xi}, \phi] \mapsto -\Phi_0 \circ \partial_{x^\xi}. \quad (3.23)$$

**Proposition 3.2.** *The image of  $\mathcal{R}_N(\mathcal{N})$  under the linear map (3.21) is the vector space  $\mathcal{N}^\bullet$ .*

*Proof.* First, let  $\phi \in \mathcal{R}_N(\mathcal{N})$  and let  $\phi \mathcal{F}_\ell \mathcal{N} \subseteq \mathcal{F}_{\ell+L} \mathcal{D}'_{\mathbb{R}^N, 0}$  ( $\ell \in \mathbb{Z}$ ). Then  $\Phi_0 \in (\mathcal{F}_\ell \mathcal{N})^\perp$  if  $\ell + L < N$ .

Conversely, let  $\Phi \in \mathcal{N}^\bullet$  and define by (3.20) and (3.22) with  $\Phi_0 = \Phi$  a linear map  $\phi : \mathcal{N} \rightarrow \mathcal{D}'_{\mathbb{R}^N, 0}$ . Note that the sum in (3.20) is always finite when we apply it on an element of  $\mathcal{N}$ , since  $\Phi \in (\mathcal{F}_\ell \mathcal{N})^\perp$  for some  $\ell \in \mathbb{Z}$  and  $\mathbf{x}^r \mathcal{F}_m \mathcal{N} \subseteq \mathcal{F}_{m-|r|} \mathcal{N}$  for every  $m \in \mathbb{Z}$ . The latter also implies that if  $\deg u = m$  then  $\text{sc.d.} \phi(u) \leq N + k$ , where  $\ell = m - k$ . Hence,  $\phi \in \mathcal{R}_N(\mathcal{N})$  since the equations  $[x^\xi, \phi] = 0$  ( $\xi = 1, \dots, N$ ) follow as above. By the construction  $\phi$  is mapped on  $\Phi$  via the assignment (3.22).  $\square$

**Corollary 3.3.** *If  $\gamma_{n;\xi}$  are primary renormalization cocycles (3.4) then the linear functionals  $\Gamma_{n;\xi}$  (3.16) belong to  $(\mathcal{O}_n)^\bullet$ . Conversely, every set of functionals  $\Gamma_{n;\xi} \in (\mathcal{O}_n)^\bullet$  determine by Eqs. (3.15), (3.16) and (3.14) a set of linear maps  $\gamma_{n;\xi} : \mathcal{O}_n \rightarrow \mathcal{D}'_{n,0}$  that satisfy the conditions  $[x^\eta, \gamma_{n;\xi}] = 0$ .*

Thus, we see that the cohomology behind the cohomological equations (3.6) is exactly the de Rham cohomology of the differential module  $(\mathcal{O}_n)^\bullet$ . Let us recall its general definition.

**Definition 3.3.** The *de Rham complex* for an arbitrary  $\mathfrak{D}_N$ -module  $\mathcal{N}$  is defined as the complex:

$$\{0\} \xrightarrow{d} \Omega^0(\mathcal{N}) \xrightarrow{d} \Omega^1(\mathcal{N}) \xrightarrow{d} \dots \xrightarrow{d} \Omega^N(\mathcal{N}) \xrightarrow{d} \{0\}, \quad (3.24)$$

where

$$\Omega^0(\mathcal{N}) \equiv \mathcal{N}, \quad \Omega^m(\mathcal{N}) := \Lambda^m(\mathbb{R}^N) \otimes \mathcal{N},$$

and  $\Lambda^m(\mathbb{R}^N)$  stands for the  $m$ th antisymmetric power of  $\mathbb{R}^N$ . Thus, the elements of  $\Omega^m(\mathcal{N})$  are represented by sequences  $\Theta = (\Theta_{\xi_1, \dots, \xi_m})$  with coefficients  $\Theta_{\xi_1, \dots, \xi_m} \in \mathcal{N}$  for  $\xi_1, \dots, \xi_m = 1, \dots, N$ , which are antisymmetric,  $\Theta_{\xi_1, \dots, \xi_m} = (-1)^{\text{sgn } \sigma} \Theta_{\xi_{\sigma_1}, \dots, \xi_{\sigma_m}}$ . The differential of  $\Theta_{\xi_1, \dots, \xi_m}$  is

$$(d\Theta)_{\xi_1, \dots, \xi_{m+1}} = \sum_{\ell=1}^m (-1)^{\ell+1} \partial_{x^{\xi_\ell}} \Theta_{\xi_1, \dots, \widehat{\xi_\ell}, \dots, \xi_{m+1}}. \quad (3.25)$$

Denote by  $H^m(\mathcal{N})$ , for  $m = 0, \dots, N$ , the cohomology group of the complex (3.24):

$$H^m(\mathcal{N}) := \mathcal{Z}^m(\mathcal{N}) / \mathcal{B}^m(\mathcal{N}),$$

$$\mathcal{Z}^m(\mathcal{N}) := \text{Ker } d \Big|_{\Omega^m(\mathcal{N})}, \quad \mathcal{B}^m(\mathcal{N}) := d(\Omega^{m-1}(\mathcal{N})).$$

For a  $\mathbb{Z}$ -filtered  $\mathfrak{D}_N$ -module  $\mathcal{N}$  we also set:

$$\begin{aligned}\mathcal{F}_\ell \Omega^m(\mathcal{N}) &:= \Lambda^m(\mathbb{R}^N) \otimes \mathcal{F}_\ell \mathcal{N}, \\ \mathcal{F}_\ell \mathcal{Z}^m(\mathcal{N}) &:= \mathcal{Z}^m(\mathcal{N}) \cap \mathcal{F}_\ell \Omega^m(\mathcal{N}), \\ \mathcal{F}_\ell \mathcal{B}^m(\mathcal{N}) &:= \mathcal{B}^m(\mathcal{N}) \cap \mathcal{F}_\ell \Omega^m(\mathcal{N})\end{aligned}$$

for  $\ell \in \mathbb{Z}$ .

Applying the above abstract results to our primary renormalization co-cycles  $\{\gamma_{n;\xi}\}_\xi$  we can claim that they are characterized by a cohomology class in  $H^1(\mathcal{O}_n^\bullet)$  ( $\mathcal{O}_n^\bullet \equiv (\mathcal{O}_n)^\bullet$ ). We can further simplify the description of this cohomology class due to the following result.

**Theorem 3.4.** *In the case of the algebra  $\mathcal{O}_n$  given by space of rational functions (2.5) there exists a natural isomorphism:*

$$H^m(\mathcal{O}_n^\bullet) \cong \left( H^{N-m}(\mathcal{O}_n) \right)'$$

for  $0 \leq m \leq N$ .

Let us point out that there is a result [1, Ch. 1, Theorems 5.5 and 6.1] stating that *all the de Rham cohomologies  $H^m(\mathcal{O}_n)$  are finite dimensional for the case when  $\mathcal{O}_n$  are given by (2.5) and hence,  $H^m(\mathcal{O}_n^\bullet)$  are finite dimensional too.*

We conclude this subsection with the *proof* of Theorem 3.4. Since it will not be used in the remaining part of this paper the reader can skip it in the first reading.

First we start with an analog of the Poincaré lemma in the case of de Rham cohomologies of  $\mathfrak{D}_N$ -modules. It uses partial inevitability of the Euler operator (vector field) on  $\mathbb{R}^N$ ,

$$\mathbf{x} \cdot \partial_{\mathbf{x}} = \sum_{\xi=1}^N x^\xi \partial_{x^\xi}.$$

Due to Eq. (3.18) this operator keeps invariant every  $\mathcal{F}_\ell \mathcal{N}$ .

**Lemma 3.5.** *Let  $\mathcal{N}$  be a  $\mathbb{Z}$ -filtered  $\mathfrak{D}_N$ -module, which is such that for some  $\ell \in \mathbb{Z}$  the operator  $k + \mathbf{x} \cdot \partial_{\mathbf{x}}$  it is invertible on  $\mathcal{F}_\ell \mathcal{N}$  for every  $k \in \mathbb{N}_0$ . Then every closed form with coefficients in  $\mathcal{F}_\ell \mathcal{N}$  is exact, i.e., if  $\Theta \in \mathcal{F}_\ell \Omega^m(\mathcal{N})$  then  $d\Theta = 0$  implies that  $\Theta = d\mathbf{B}$  for some  $\mathbf{B} \in \Omega^{m-1}(\mathcal{N})$ .*

*Proof.* The proposition can be proven by using the  $\mathfrak{D}_N$ -analog of the Poincaré homotopy operator

$$(K\Theta)_{\xi_1, \dots, \xi_{m-1}} = (m-1 + \mathbf{x} \cdot \partial_{\mathbf{x}})^{-1} \sum_{\xi=1}^N x^\xi \Theta_{\xi, \xi_1, \dots, \xi_{m-1}}$$

where  $\Theta = (\Theta_{\xi_1, \dots, \xi_m}) \in \mathcal{F}_\ell \Omega^m(\mathcal{N}) = \Lambda^m(\mathbb{R}^N) \otimes \mathcal{F}_\ell \mathcal{N}$ . Since the operators  $k + \mathbf{x} \cdot \partial_{\mathbf{x}}$  commute with the derivatives  $\partial_{x^\xi}$  then the operators  $(k + \mathbf{x} \cdot \partial_{\mathbf{x}})^{-1}$  restricted to  $\mathcal{F}_{\ell-1} \mathcal{N}$  and  $\mathcal{F}_\ell \mathcal{N}$  also commute with  $\partial_{x^\xi} : \mathcal{F}_{\ell-1} \mathcal{N} \rightarrow \mathcal{F}_\ell \mathcal{N}$ . Hence, we derive the identity  $Kd + dK = \text{id}$  from which the proposition follows.  $\square$

We continue with the *proof of Theorem 3.4*. There is a natural pairing

$$\Omega^m(\mathcal{N}^\bullet) \otimes \Omega^{N-m}(\mathcal{N}) \rightarrow \mathbb{R} : (\Omega, \alpha) \mapsto \Omega^\wedge[\alpha]$$

for every  $m = 0, \dots, N$ , where  $\Omega^\wedge[\alpha]$  means the action of  $\Omega$  as a linear functional under the external product  $\wedge$ . Precisely, for  $\Omega = a \otimes \Phi \in \Lambda^m(\mathbb{R}^N) \otimes \mathcal{N}^\bullet$  and  $\alpha = b \otimes f \in \Lambda^{N-m}(\mathbb{R}^N) \otimes \mathcal{N}$  we set:

$$\Omega^\wedge[\alpha] := (a \wedge b) \Phi[f],$$

where  $a \wedge b$  is considered as an element of  $\mathbb{R} \cong \Lambda^N(\mathbb{R}^N)$ . Then we have

$$(d\Omega)^\wedge[\alpha] = (-1)^{m+1} \Omega^\wedge[d\alpha] \quad (3.26)$$

for  $\Omega \in \Omega^m(\mathcal{N})$  and  $\alpha \in \Omega^{N-m-1}(\mathcal{N})$ , according to Eq. (3.25). We denote

$$\Omega^\wedge : \Omega^{N-m}(\mathcal{N}) \rightarrow \mathbb{R} : \alpha \mapsto \Omega^\wedge[\alpha].$$

Note now that if  $\Omega$  is closed then  $\Omega^\wedge[d\alpha'] = 0$  for every  $\alpha' \in \Omega^{N-m-1}(\mathcal{N})$  and if  $\Omega$  is exact then  $\Omega^\wedge[\alpha] = 0$  for every closed  $\alpha \in \Omega^{N-m}(\mathcal{N})$ . Hence, for closed  $\Omega$  and  $\alpha$  the number  $\Omega^\wedge[\alpha]$  depends only on the cohomology classes of  $\Omega$  and  $\alpha$  and thus we obtain a natural linear map

$$H^m(\mathcal{N}^\bullet) \rightarrow \left( H^{N-m}(\mathcal{N}) \right)'. \quad (3.27)$$

**Lemma 3.6.** *For a  $\mathfrak{D}_N$ -module  $\mathcal{N}$  satisfying the conditions of Lemma 3.5 the above natural map (3.27) is an isomorphism.*

*Proof.* First we prove that (3.27) is surjective. Let us have a linear functional

$$\mathcal{Z}^{N-m}(\mathcal{N}) / \mathcal{B}^{N-m}(\mathcal{N}) (= H^{N-m}(\mathcal{N})) \xrightarrow{\Omega'} \mathbb{R}. \quad (3.28)$$

Then our task is to extend  $\Omega'$  to a linear functional

$$\Omega^{N-m}(\mathcal{N}) / \mathcal{B}^{N-m}(\mathcal{N}) \xrightarrow{\Omega''} \mathbb{R} \quad (3.29)$$

such that  $\Omega'' \circ \pi = \Omega^\wedge$  for an element  $\Omega \in \Omega^m(\mathcal{N}^\bullet)$ , where  $\pi$  is the natural projection  $\Omega^{N-m}(\mathcal{N}) \rightarrow \Omega^{N-m}(\mathcal{N}) / \mathcal{B}^{N-m}(\mathcal{N})$ . It is always possible to extend  $\Omega'$  (3.28) to some linear functional  $\Omega''$  (3.29) and we point out also that every linear functional  $\Theta : \Omega^{N-m}(\mathcal{N}) \rightarrow \mathbb{R}$  is of a form  $\Theta^\wedge$  for a unique  $\Theta \in \Omega^m(\mathcal{N}')$ . So, we have an element  $\Omega \in \Omega^m(\mathcal{N}')$  such that  $\Omega^\wedge = \Omega'' \circ \pi$  and it remains only to achieve  $\Omega \in \Omega^m(\mathcal{N}^\bullet)$ . The latter requires to impose further conditions on the extension  $\Omega''$  (3.29). We require that  $\Omega''$  is zero on

$$\pi(\mathcal{F}_\ell \Omega^{N-m}(\mathcal{N})) \equiv \mathcal{F}_\ell \Omega^{N-m}(\mathcal{N}) / \mathcal{B}^{N-m}(\mathcal{N})$$

for some  $\ell \in \mathbb{Z}$ . This is always possible since

$$\begin{aligned} & \left( \mathcal{F}_\ell \Omega^{N-m}(\mathcal{N}) / \mathcal{B}^{N-m}(\mathcal{N}) \right) \cap \left( \mathcal{Z}^{N-m}(\mathcal{N}) / \mathcal{B}^{N-m}(\mathcal{N}) \right) \\ &= \mathcal{F}_\ell \mathcal{Z}^{N-m}(\mathcal{N}) / \mathcal{B}^{N-m}(\mathcal{N}) = \{0\} \end{aligned}$$

for  $\ell$  chosen according to the assumptions of Proposition 3.5. In this way we have  $\Omega^\wedge[\alpha] = 0$  if  $\alpha \in \mathcal{F}_\ell \Omega^{N-m}(\mathcal{N})$  and hence,  $\Omega \in \Omega^m(\mathcal{N}^\bullet)$ . Thus, the linear map (3.27) is surjective.

To prove that the map (3.27) is injective assume that  $\Omega \in \Omega^m(\mathcal{N}^\bullet)$  is such that  $\Omega^\wedge[\alpha] = 0$  for all  $\alpha \in \mathcal{Z}^{N-m}(\mathcal{N})$ . We should prove that  $\Omega = d\Theta$  for  $\Theta \in \Omega^{m-1}(\mathcal{N}^\bullet)$ . To this end we note first that  $\Omega^\wedge|_{\mathcal{F}_{\ell'} \Omega^{N-m}(\mathcal{N})} = 0$  for some  $\ell' \in \mathbb{Z}$  and we set  $\ell_0 = \min\{\ell, \ell'\}$ , where  $\ell \in \mathbb{Z}$  is the integer from the assumptions of Proposition 3.5. Thus,  $\Omega^\wedge|_{\mathcal{F}_{\ell_0} \Omega^{N-m}(\mathcal{N})} = 0$ . Now consider the short exact sequence

$$\begin{aligned} 0 \rightarrow \mathcal{Z}^{N-m}(\mathcal{N}) / \mathcal{F}_{\ell_0-1} \mathcal{Z}^{N-m}(\mathcal{N}) &\hookrightarrow \Omega^{N-m}(\mathcal{N}) / \mathcal{F}_{\ell_0-1} \Omega^{N-m}(\mathcal{N}) \\ &\xrightarrow{d} \mathcal{B}^{N-m+1}(\mathcal{N}) / \mathcal{F}_{\ell_0} \mathcal{B}^{N-m+1}(\mathcal{N}) \rightarrow 0, \end{aligned}$$

which is due to  $\mathcal{F}_{\ell_0} \mathcal{B}^{N-m+1}(\mathcal{N}) = d \mathcal{F}_{\ell_0-1} \Omega^{N-m}(\mathcal{N})$  (Proposition 3.5). Then we obtain a linear functional

$$\Theta' : \mathcal{B}^{N-m+1}(\mathcal{N}) / \mathcal{F}_{\ell_0} \mathcal{B}^{N-m+1}(\mathcal{N}) \rightarrow \mathbb{R}$$

such that  $\Omega^\wedge = \Theta' \circ \pi' \circ d$ , where  $\pi'$  is the projection

$$\mathcal{B}^{N-m+1}(\mathcal{N}) \xrightarrow{\pi'} \mathcal{B}^{N-m+1}(\mathcal{N}) / \mathcal{F}_{\ell_0} \mathcal{B}^{N-m+1}(\mathcal{N}).$$



Finally, we extend  $\Theta'$  to a linear functional  $\Theta''$ ,

$$\Theta'' : \Omega^{N-m+1}(\mathcal{N}) / \mathcal{F}_{\ell_0} \Omega^{N-m+1}(\mathcal{N}) \rightarrow \mathbb{R}$$

(under the natural embedding

$$\mathcal{B}^{N-m+1}(\mathcal{N}) / \mathcal{F}_{\ell_0} \mathcal{B}^{N-m+1}(\mathcal{N}) \hookrightarrow \Omega^{N-m+1}(\mathcal{N}) / \mathcal{F}_{\ell_0} \Omega^{N-m+1}(\mathcal{N}))$$

and setting  $\Theta^\wedge := \Theta'' \circ \pi''$ , where

$$\Omega^{N-m+1}(\mathcal{N}) \xrightarrow{\pi''} \Omega^{N-m+1}(\mathcal{N}) / \mathcal{F}_{\ell_0} \Omega^{N-m+1}(\mathcal{N}),$$

we get by Eq. (3.26) that  $\Omega = (-1)^{m+1} d\Theta$  and  $\Theta \in \Omega^{m-1}(\mathcal{N}^\bullet)$  (i.e.,  $\Omega^\wedge[\alpha] = \Theta^\wedge[d\alpha]$  for all  $\alpha \in \Omega^{m-1}(\mathcal{N})$ ). Hence,  $\Omega$  is exact and thus, the linear map (3.27) is also injective.  $\square$

To complete the proof of Theorem 3.4 we need to apply Lemma 3.6 to the  $\mathfrak{D}_{D(n-1)}$ -module  $\mathcal{O}_n$  (2.5). This is possible due to the following result.

**Lemma 3.7.** *The  $\mathfrak{D}_{D(n-1)}$ -modules  $\mathcal{O}_n$  (2.5) satisfy the assumptions of Lemma 3.5.*

*Proof.* We shall prove that all the operators  $\ell + \mathbf{x} \cdot \partial_{\mathbf{x}}$ , for  $\ell = 0, 1, 2, \dots$ , are invertible on the subspace of  $\Omega^m(\mathcal{O}_n)$ , which consists of elements with negative scaling degree. Indeed, if  $\Theta = (\Theta_{\xi_1, \dots, \xi_m}(\mathbf{x})) \in \Omega^m(\mathcal{O}_n)$  and  $\deg \Theta < 0$  then for every  $\mathbf{x} \notin \widehat{\Delta}_n$  ( $:= \mathbf{E}_n \setminus \mathbf{F}_n$ ) the function  $\lambda^{-1} \Theta_{\xi_1, \dots, \xi_m}(\lambda \mathbf{x})$  is integrable for  $\lambda \in (0, 1)$ . Hence, we define  $(\ell + \mathbf{x} \cdot \partial_{\mathbf{x}})^{-1} \Theta$  by

$$\left( (\ell + \mathbf{x} \cdot \partial_{\mathbf{x}})^{-1} \Theta \right)_{\xi_1, \dots, \xi_m}(\mathbf{x}) = \int_0^1 \lambda^{\ell-1} \Theta_{\xi_1, \dots, \xi_m}(\lambda \mathbf{x}) d\lambda.$$

The right hand side above defines an element of  $\mathcal{O}_n$  since the integrand is a polynomial in  $\lambda$  with coefficients in  $\mathcal{O}_n$ .  $\square$

### 3.3 Reduction of the cohomological equations

By the results of the previous subsections we have characterized the primary renormalization cocycles  $\{\gamma_{n;\xi}\}_\xi$ , for every  $n = 2, 3, \dots$  by 1-forms

$$\Gamma_n = \sum_{\xi=1}^N \Gamma_{n;\xi} dx^\xi \in \Omega^1(\mathcal{O}_n^\bullet)$$

( $N = D(n-1)$ ,  $\mathcal{O}_n^\bullet \equiv (\mathcal{O}_n)^\bullet$ ). Then  $\gamma_{n;\xi}$  are constructed by the formula:

$$\gamma_{n;\xi}(G) = \sum_{\mathbf{r} \in \mathbb{N}_0^N} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!} \Gamma_{n;\xi}(\mathbf{x}^{\mathbf{r}} G) \delta^{(\mathbf{r})}(\mathbf{x})$$

( $G \in \mathcal{O}_n$ ). On the other hand,  $\{\gamma_{n;\xi}\}_\xi$  satisfy the cohomological equations (3.6) and we argued in Sect. 3.1 that their right hand sides depend on renormalization maps of lower order. Thus, one can expect that these equations are equivalent to some equations for  $\mathbf{\Gamma}_n$  of a form:

$$\begin{aligned} d\mathbf{\Gamma}_2 &= 0, \\ d\mathbf{\Gamma}_n &= \mathcal{F}_n[\mathbf{\Gamma}_1, \dots, \mathbf{\Gamma}_{n-1}] \quad (n > 2). \end{aligned} \quad (3.30)$$

In order to derive the right hand side of (3.30) in a simple explicit form we shall need some notations. Similarly to Sect. 2 we introduce set-dependent notations:

$$\begin{aligned} \gamma_{S;\xi} &:= [\partial_{x^\xi}, P_S] \circ \dot{R}_S = \sum_{\mathbf{r} \in \mathbb{N}_0^N} \frac{(-1)^{|\mathbf{r}|}}{\mathbf{r}!} \delta_S^{(\mathbf{r})} \Gamma_{S;\xi} \circ \mathbf{x}^{\mathbf{r}}, \\ \mathbf{\Gamma}_S &:= \sum_{\xi=1}^N \Gamma_{S;\xi} dx^\xi \in \Omega^1(\mathcal{O}_S^\bullet). \end{aligned}$$

( $N = D(|S|-1)$ ) for every finite subset  $S \subset \mathbb{N}$  with at least two elements. In the above equations we assume that we have fixed some (linear) coordinates on  $\mathbf{E}_S$ , which we denote by  $x^\xi$  ( $\xi = 1, \dots, N$ ).

For  $S' \subseteq S$  ( $\subseteq_{\text{fin.}} \mathbb{N}$ ) we set

$$S/S' := (S \setminus S') \cup \{\min S'\}$$

and for every two subsets  $S', S'' \subseteq S$  such that  $S' \cup S'' = S$  and  $|S' \cap S''| = 1$  (for instance  $S'$  and  $S'' = S/S'$  is such a pair) there is a canonical isomorphism

$$\mathbf{E}_S \cong \mathbf{E}_{S'} \oplus \mathbf{E}_{S''} : [\mathbf{x}_j]_{j \in S} \mapsto ([\mathbf{x}_{j'}]_{j' \in S'}, [\mathbf{x}_{j''}]_{j'' \in S''}) \quad (3.31)$$

(recall that  $\mathbf{E}_S := (\mathbb{R}^D)^S / \mathbb{R}^D$  and its elements, denoted by  $[\mathbf{x}_j]_{j \in S}$ , are equivalence classes of elements  $(\mathbf{x}_j)_{j \in S} \in (\mathbb{R}^D)^S$ ). Let us consider the pair  $(S', S'' = S/S')$  for a nonempty  $S' \subsetneq S$  and we assume that we have equipped  $\mathbf{E}_{S'}$  with (linear) coordinates  $\mathbf{x}' = (x'^{\xi'})$  for  $\xi' = 1, \dots, N'$  ( $N' = D(|S'| -$

1)) and  $\mathbf{E}_{S/S'}$  with coordinates  $\mathbf{x}'' = (x''^{\xi''})$  for  $\xi'' = 1, \dots, N''$  ( $N'' = D(|S/S'| - 1) = D(|S| - |S'|)$ ). Then we denote  $(\Pi'_{S'})_{\xi}^{\xi'} := \frac{\partial x'^{\xi'}}{\partial x^{\xi}}$  and  $(\Pi''_{S'})_{\xi}^{\xi''} := \frac{\partial x''^{\xi''}}{\partial x^{\xi}}$  ( $\xi = 1, \dots, N$ ,  $\xi' = 1, \dots, N'$ ,  $\xi'' = 1, \dots, N''$ ) using the change of coordinates  $(x^{\xi}) \mapsto (x'^{\xi'}, x''^{\xi''})$  implied by Eq. (3.31).

Using the above notations we introduce bilinear operations (see Lemma 3.8)

$$\overset{\circ}{\wedge} : \Omega^k(\mathcal{O}_{n-m+1}^{\bullet}) \otimes \Omega^{\ell}(\mathcal{O}_m^{\bullet}) \rightarrow \Omega^{k+\ell}(\mathcal{O}_n^{\bullet}) \quad (3.32)$$

for every  $1 < m < n$  in the following way. Set  $S := \{1, \dots, n\}$  and for every  $\Theta \in \Omega^k(\mathcal{O}_m^{\bullet})$  and  $S' \subset S$  with  $|S'| = m$  we denote by  $\Theta_S$  the natural lift of  $\Theta$  from  $\Omega^k(\mathcal{O}_m^{\bullet})$  to  $\Omega^k(\mathcal{O}_{S'}^{\bullet})$  under the bijection  $\{1, \dots, m\} \cong S' : s \mapsto j_s$  for  $j_1 < \dots < j_s$ . Then for  $\Theta' \in \Omega^k(\mathcal{O}_m^{\bullet})$  and  $\Theta'' \in \Omega^{\ell}(\mathcal{O}_{n-m+1}^{\bullet})$  we set

$$\begin{aligned} & (\Theta'' \overset{\circ}{\wedge} \Theta')(G_S) \\ &= \sum_{\substack{S' \subsetneq S \\ |S'|=m}} \sum_{\mathbf{r}' \in \mathbb{N}_0^{N'}} \frac{1}{\mathbf{r}'!} (\Pi''_{S'})^* \Theta''_{S/S'} \left( \partial_{\mathbf{x}'}^{\mathbf{r}'} G_{\mathfrak{P}(S')} \Big|_{\mathbf{x}'=0} \right) \\ & \quad \wedge (\Pi'_{S'})^* \Theta'_{S'} (\mathbf{x}'^{\mathbf{r}'} G_{S'}), \end{aligned} \quad (3.33)$$

where  $G_S \in \mathcal{O}_S$  is taken of the form (2.14),

$$G_S = G_{\mathfrak{P}(S')} G_{S'}, \quad (3.34)$$

for the  $S$ -partition  $\mathfrak{P}(S') := \{\{j\} : j \in S \setminus S'\} \cup \{S'\}$ ,  $(\Pi'_{S'})^*$  and  $(\Pi''_{S'})^*$  stand for the obvious pullbacks, e.g.,

$$\begin{aligned} & (\Pi'_{S'})^* : \Lambda^k(\mathbf{E}_{S'}) \rightarrow \Lambda^k(\mathbf{E}_S) \\ & : \lambda_{\xi'_1, \dots, \xi'_k} \mapsto \sum_{\xi'_1, \dots, \xi'_k=1}^{N'} (\Pi_{S'})_{\xi'_1}^{\xi'_1} \cdots (\Pi_{S'})_{\xi'_k}^{\xi'_k} \lambda_{\xi'_1, \dots, \xi'_k} \end{aligned}$$

and finally,  $\Theta(G)$  for  $\Theta \in \Omega^r(\mathcal{O}_s^{\bullet})$  and  $G \in \mathcal{O}_s$  is considered as an element of  $\Lambda^r(\mathbf{E}_s)$ .

**Lemma 3.8.** *The bilinear operation  $\overset{\circ}{\wedge}$  (3.32) is well defined by Eq. (3.33).*

*Proof.* First we point out that the sum in the right hand side of Eq. (3.33) is always finite since the scaling degree of  $\mathbf{x}'^{\mathbf{r}'} G_{S'}$  decreases as  $|\mathbf{r}'|$  increases. Next, we need to show that Eq. (3.33) defines a linear map in  $G \in \mathcal{O}_n$ . This follows from the fact that for every  $S' \subsetneq S$  the assignment

$$G_S \mapsto \sum_{\mathbf{r}' \in \mathbb{N}_0^{N'}} \frac{1}{\mathbf{r}'!} \left( \partial_{\mathbf{x}'}^{\mathbf{r}'} G_{\mathfrak{P}(S')} \Big|_{\mathbf{x}'=0} \right) \cdot (\mathbf{x}'^{\mathbf{r}'} G_{S'})$$

extends to a linear map in  $G \in \mathcal{O}_n$  since this is just the Taylor expansion of  $G_{\mathfrak{P}(S')}$  in  $\mathbf{x}'$ . Finally, it is not difficult to show that the so defined  $\Theta'' \overset{\circ}{\wedge} \Theta'$  indeed belongs to  $\Omega^{k+\ell}(\mathcal{O}_n^\bullet)$ , i.e., it gives zero on all functions  $G_S$  belonging to  $\mathcal{F}_L \mathcal{O}_n$  for some  $L \in \mathbb{Z}$  (cf. Eq. 3.19).  $\square$

The operation  $\overset{\circ}{\wedge}$  (3.32) is not  $\mathbb{Z}/2\mathbb{Z}$ -symmetric. Under the above notations we have the following result:

**Theorem 3.9.** *The cohomological equations (3.6) for  $\{\gamma_{n;\xi}\}_\xi$  are equivalent to the following equations for  $\Gamma_n$ :*

$$\begin{aligned} d\Gamma_2 &= 0, \\ d\Gamma_n &= \sum_{m=2}^{n-1} \Gamma_{n-m+1} \overset{\circ}{\wedge} \Gamma_m \quad (n > 2). \end{aligned} \quad (3.35)$$

For the *proof* of this theorem we shall also use some notations used in Sect. 2.6: every function  $G_{\mathfrak{P}(S')} u_{S'} \in \mathcal{O}_{\mathfrak{P}(S')} \otimes \mathcal{D}'_{S',0}$  can be uniquely transformed to a function belonging to  $\mathcal{O}_{S/S'} \otimes \mathcal{D}'_{S',0}$ , which includes a restriction to the partial diagonal  $\{x_{\min S'} = x_j \text{ for all } j \in S'\}$  (with possible “transverse” derivatives due to the possible derivatives of the delta functions contained in  $u_{S'}$ ). The latter transformation we denote by  $n.f._{S'}$  (“normal form” corresponding to  $n.f._{\mathfrak{P}(S')}$  used in Sect. 2.6):

$$n.f._{S'} : \mathcal{O}_{\mathfrak{P}(S')} \otimes \mathcal{D}'_{S',0} \rightarrow \mathcal{O}_{S/S'} \otimes \mathcal{D}'_{S',0}.$$

**Lemma 3.10.** *Let  $|S| \geq 3$ ,  $N = D(|S|-1)$ ,  $N' = D(|S'|-1)$  for  $S' \subseteq S$ . For every  $\xi = 1, \dots, N$  and  $G_S \in \mathcal{O}_S$  of the form (3.34) we have the following identities*

$$\begin{aligned} & [\partial_{x^\xi}, \dot{R}_S] G_S \\ &= \sum_{\substack{S' \subsetneq S \\ |S'| \geq 2}} \left( \dot{R}_{S/S'} \otimes id_{\mathcal{D}'_{S',0}} \right) \circ n.f._{S'} \left( G_{\mathfrak{P}(S')} \cdot \sum_{\xi'=1}^{N'} (\Pi_{S'})_{\xi}^{\xi'} \gamma_{S';\xi'}(G_{S'}) \right). \end{aligned} \quad (3.36)$$

*Proof.* Due to the covering property (2.18) it is enough to prove that

$$\text{left hand side of Eq. (3.36)}|_{\mathbf{F}_{\mathfrak{P}}} = \text{right hand side of Eq. (3.36)}|_{\mathbf{F}_{\mathfrak{P}}} \quad (3.37)$$

for every proper  $S$ -partition  $\mathfrak{P}$ . For the restriction of the right hand side we obtain:

$$\sum_{\substack{S' \leq \mathfrak{P} \\ |S'| \geq 2}} \left( \dot{R}_{S/S'} \otimes id_{\mathcal{D}_{S',0}'} \right) \circ n.f.S' \left( G_{\mathfrak{P}(S')} \cdot \sum_{\xi'=1}^{N'} (\Pi_{S'})_{\xi'}^{\xi'} \gamma_{S';\xi'}(G_{S'}) \right) \Big|_{\mathbf{F}_{\mathfrak{P}}}, \quad (3.38)$$

where the notation  $S' \leq \mathfrak{P}$  stands for the relation:  $j, k \in S' \Rightarrow j \sim_{\mathfrak{P}} k$ . This is because if  $S' \not\leq \mathfrak{P}$  then the support of  $\gamma_{S';\xi'} G_{S'}$  is disjoint from  $\mathbf{F}_{\mathfrak{P}}$ . For the restriction of the left hand side of Eq. (3.36) we obtain from condition (r4) that:

$$\begin{aligned} & [\partial_{x^\xi}, \dot{R}_S] G_S \Big|_{\mathbf{F}_{\mathfrak{P}}} \\ &= G_{\mathfrak{P}} \cdot \left( \sum_{S' \in \mathfrak{P}} \sum_{\xi'=1}^{N'} (\Pi_{S'})_{\xi'}^{\xi'} [\partial_{x'^{\xi'}}, R_{S'}](G_{S'}) \right) \prod_{\substack{S'' \in \mathfrak{P} \\ S'' \neq S'}} R_{S''} G_{S''}. \end{aligned} \quad (3.39)$$

We then use the identities

$$[\partial_{x'^{\xi'}}, R_{S'}](G_{S'}) = \gamma_{S';\xi'}(G_{S'}) \quad \text{for } |S'| = 2, \quad (3.40)$$

$$[\partial_{x'^{\xi'}}, R_{S'}](G_{S'}) = \gamma_{S';\xi'}(G_{S'}) + P_{S'} \circ [\partial_{x'^{\xi'}}, \dot{R}_{S'}](G_{S'}) \quad \text{for } |S'| > 2. \quad (3.41)$$

From the first of these identities and Eq. (3.38) we obtain Eq. (3.37) for the case  $|S| = 3$ .

For  $|S| > 3$  we proceed by induction in  $|S|$  and apply the inductive assumption to the second term in the right hand side of Eq. (3.41). In this way, substituting the result in Eq. (3.39) we arrive at the expression (3.38) by using the properties of renormalization maps and in particular, (p5). Thus, we obtain again Eq. (3.37), which proves the lemma.  $\square$

Now to complete the proof of Theorem 3.9 we need only to compose both sides of Eq. (3.36) with  $[\partial_{x^\eta}, P_S]$  and use property (p5) of Sect. (2.4) in order to reduce  $[\partial_{x^\eta}, P_S]$  to  $\sum_{\eta''=1}^{N''} (\Pi_{S'}^{\eta''})_{\eta''}^{\eta''} [\partial_{x''\eta''}, P_{S/S'}]$ . In this way

we obtain:

$$\begin{aligned}
 & [\partial_{x^\eta}, P_S] \circ [\partial_{x^\xi}, \dot{R}_S] G_S \\
 &= \sum_{\substack{S' \subsetneq S \\ |S'| \geq 2}} \left( \sum_{\eta''=1}^{N''} (\Pi_{S'}'')_{\eta}^{\eta''} \gamma_{S/S'; \eta''} \otimes id_{\mathcal{D}'_{S',0}} \right) \\
 &\quad \circ n.f.S' \left( G_{\mathfrak{P}(S')} \cdot \sum_{\xi'=1}^{N'} (\Pi_{S'})_{\xi}^{\xi'} \gamma_{S'; \xi'}(G_{S'}) \right) \\
 &= \sum_{\substack{S' \subsetneq S \\ |S'| \geq 2}} \sum_{\xi', \eta''} (\Pi_{S'}'')_{\eta}^{\eta''} (\Pi_{S'})_{\xi}^{\xi'} \left( \gamma_{S/S'; \eta''} \otimes id_{\mathcal{D}'_{S',0}} \right) \\
 &\quad \circ n.f.S' \left( G_{\mathfrak{P}(S')} \cdot \sum_{\mathbf{r}' \in \mathbb{N}_0^{N'}} \frac{(-1)^{|\mathbf{r}'|}}{\mathbf{r}'!} \Gamma_{S'; \xi'}(\mathbf{x}'^{\mathbf{r}'} G_{S'}) \delta^{(\mathbf{r}')}(\mathbf{x}') \right) \\
 &= \sum_{\substack{S' \subsetneq S \\ |S'| \geq 2}} \sum_{\xi', \eta''} (\Pi_{S'}'')_{\eta}^{\eta''} (\Pi_{S'})_{\xi}^{\xi'} \sum_{\mathbf{r}' \in \mathbb{N}_0^{N'}} \frac{1}{\mathbf{r}'!} \Gamma_{S/S'; \eta''} \left( \partial_{\mathbf{x}'}^{\mathbf{r}'} G_{\mathfrak{P}(S')} \Big|_{\mathbf{x}'=0} \right) \\
 &\quad \times \Gamma_{S'; \xi'}(\mathbf{x}'^{\mathbf{r}'} G_{S'}) \cdot \delta(\mathbf{x}) + \dots,
 \end{aligned} \tag{3.42}$$

where in the last line the dots include terms with derivatives of the delta function  $\delta(\mathbf{x})$ . In this way we arrive to Eqs. (3.35) and we note that there is a change in the sign due to the passage in the left hand side of Eqs. (3.6) from  $[\partial_{x^\xi}, \gamma_{n; \eta}]$  to  $\partial_{x^\xi} \Gamma_{n; \eta}$  in accordance to Eq. (3.23). This completes the proof of Theorem 3.9.

**Remark 3.1.** An element  $\Theta \in \mathcal{O}_m^\bullet$  is called symmetric iff  $\Theta \circ \sigma^* = \Theta$  for every  $\sigma \in \mathcal{S}_m$  ( $(\sigma^* F)(x_{j_1}, \dots, x_{j_n}) := F(x_{\sigma(j_1)}, \dots, x_{\sigma(j_n)})$ ). An element  $\Theta \in \Omega^k(\mathcal{O}_m^\bullet)$  is called symmetric iff all its components  $\Theta_{\xi_1, \dots, \xi_m}$  are symmetric elements of  $\mathcal{O}_m^\bullet$ . One can prove that if  $\Theta \in \Omega^k(\mathcal{O}_m^\bullet)$  and  $\Theta' \in \Omega^{k'}(\mathcal{O}_{m'}^\bullet)$  are symmetric then so is  $\Theta \overset{\circ}{\wedge} \Theta'$ . Furthermore, the bilinear operations  $\overset{\circ}{\wedge}$  (3.32) are associative on symmetric elements in the sense that:

$$\Theta \overset{\circ}{\wedge} (\Theta' \overset{\circ}{\wedge} \Theta'') = (\Theta \overset{\circ}{\wedge} \Theta') \overset{\circ}{\wedge} \Theta''$$

for  $\Theta \in \Omega^k(\mathcal{O}_m^\bullet)$ ,  $\Theta' \in \Omega^{k'}(\mathcal{O}_{m'}^\bullet)$  and  $\Theta'' \in \Omega^{k''}(\mathcal{O}_{m''}^\bullet)$  (all being symmetric). We also claim (without a proof) that  $\overset{\circ}{\wedge}$  obeys the  $\mathbb{Z}/2\mathbb{Z}$ -graded Leibnitz rule:

$$d(\Theta \overset{\circ}{\wedge} \Theta') = (d\Theta) \overset{\circ}{\wedge} \Theta' + (-1)^{m'} \Theta \overset{\circ}{\wedge} (d\Theta').$$

In this way, we obtain the integrability condition for Eqs. (3.35): if  $\Gamma_m$  are solutions of Eqs. (3.35) for  $m = 2, \dots, n-1$  then the right hand side of the second of Eqs. (3.35) is closed. One can also make the direct sum

$$\Omega_{\mathcal{O}} = \bigoplus_{m=0}^{\infty} (\Omega_{\mathcal{O}})_m, \quad (\Omega_{\mathcal{O}})_0 := \kappa, \quad (\Omega_{\mathcal{O}})_m := (\Omega^*(\mathcal{O}_m^\bullet))^{\text{symm}} \quad \text{for } m > 0$$

into a graded associative differential noncommutative algebra in which Eqs. (3.35) simply read:

$$d\mathbf{\Gamma} = \mathbf{\Gamma} \overset{\circ}{\wedge} \mathbf{\Gamma},$$

where  $(\mathbf{\Gamma})_0 = 0$  and  $(\mathbf{\Gamma})_n = \mathbf{\Gamma}_{n+1}$  for  $n > 0$ .

**Remark 3.2.** At the end of Sect. 2.6 we have introduced a notion of “universal renormalization group” with a product given by Eq. (2.48). The elements of this group can be presented by sets  $\underline{Q} = (Q_n)_{n=1}^{\infty}$  and we shall lift the grading, as in the previous remark, by setting  $(\underline{Q})_n := Q_{n+1}$  for  $n = 0, 1, \dots$ . Note that  $(\underline{Q})_n$  for  $n > 0$  is a linear map  $\mathcal{O}_{n+1}^\bullet \rightarrow \mathcal{D}'_{n+1,0}$ , which commutes with the multiplication by polynomials. Hence, expanding it in delta functions and derivatives, as in Sect. 3.2, we can further reduce its description to a linear functional belonging to  $\mathcal{O}_{n+1}^\bullet$ . In this way, the Lie algebra corresponding to the universal renormalization group will be formed by sets  $\underline{\Theta} = ((\Theta)_n)_{n=1}^{\infty}$  such that  $(\Theta)_0 = 0$  and  $(\Theta)_n \in \mathcal{O}_{n+1}^\bullet$  are symmetric (see the previous remark) for  $n = 1, 2, \dots$ . The Lie algebra bracket is very close to the product  $\overset{\circ}{\wedge}$  (3.32) and it is just the commutator

$$[\underline{\Theta}', \underline{\Theta}'] = \underline{\Theta}' \bullet \underline{\Theta}'' - \underline{\Theta}'' \bullet \underline{\Theta}',$$

where  $\bullet : \mathcal{O}_{n'+1}^\bullet \otimes \mathcal{O}_{n''+1}^\bullet \rightarrow \mathcal{O}_{n'+n''+1}^\bullet$  are associative bilinear operations defined by

$$\begin{aligned} & (\underline{\Theta}'' \bullet \underline{\Theta}')_{n-1}(G_S) \\ &= \sum_{\emptyset \subsetneq S' \subsetneq S} \sum_{\mathbf{r}' \in \mathbb{N}_0^{N'}} \frac{1}{\mathbf{r}'!} \underline{\Theta}''_{S/S'} \left( \partial_{\mathbf{x}'}^{\mathbf{r}'} G_{\mathfrak{P}(S')} \Big|_{\mathbf{x}'=0} \right) \underline{\Theta}'_{S'}(\mathbf{x}'^{\mathbf{r}'} G_{S'}) \end{aligned} \quad (3.43)$$

(we use the same type of notations like in Eq. (3.33)).

### 3.4 Concluding remarks

We intend to study the cohomological equations (3.35) and their solutions in the future. On  $\mathcal{O}_n$  the cohomological equations do not completely characterize the renormalization cocycles since, as we have pointed out, we have

$$H^1(\mathcal{O}_n^\bullet) \cong \left( H^{D(n-1)-1}(\mathcal{O}_n) \right)' \neq \{0\}. \quad (3.44)$$

In fact,  $H^m(\mathcal{O}_n)$  are the cohomology groups of the complement of union of quadrics in  $\mathbb{C}^{D(n-1)}$ :  $x_k^2 = 0 = (x_j - x_k)^2$  ( $j, k = 1, \dots, n-1$ ). Equation (3.44) is exactly the reason for which we need to introduce some transcendental methods in order to derive the renormalization cocycles. This is because otherwise, all the solutions of the cohomological equations would correspond to some renormalization scheme and hence, there will be no need to extend the ground field  $\mathbb{C}$  ( $= \mathbb{Q}$ ).

Let us point out that if we had used instead of  $\mathcal{O}_n$  the algebra  $\mathcal{C}_{temp}^\infty(\mathbf{F}_n)$  then the primary renormalization cocycles would belong to  $\Omega^1(\mathcal{C}_{temp}^\infty(\mathbf{F}_n)^\bullet)$ , whose cohomology group is now isomorphic to  $(H^{D(n-1)-1}(\mathcal{C}_{temp}^\infty(\mathbf{F}_n)))'$ . One can show that  $H^{D(n-1)-1}(\mathcal{C}_{temp}^\infty(\mathbf{F}_n))$  is isomorphic to the usual de Rham cohomology group  $H^{D(n-1)-1}(\mathbf{F}_n)$  of the configuration space  $\mathbf{F}_n$ . On the other hand, there is a theorem [8] stating that for  $n > 2$ :  $H^{D(n-1)-1}(\mathbf{F}_n) = \{0\}$  (recall that  $\mathbf{F}_n \cong F_{n-1}(\mathbb{R}^D \setminus \{\mathbf{0}\})$ ). Thus, in this case the cohomological equations would completely characterize the renormalization cocycles for  $n > 2$ . In the preprint [14, Sects. 2 and 3] we have considered renormalization on  $\mathcal{C}_{temp}^\infty(\mathbf{F}_n)$  exactly for this purpose. The problem then is that the right hand side of the cohomological equations (3.35) do not have a simple, algebraic construction. The space  $\mathcal{C}_{temp}^\infty(\mathbf{F}_n)$  is too large to work on it “algebraically”. So, the problem is to find an intermediate differential extension

$$\mathcal{O}_n \subseteq \tilde{\mathcal{O}}_n \subseteq \mathcal{C}_{temp}^\infty(\mathbf{F}_n)$$

for every  $n > 2$  such that first, it possesses an algebraic interpretation of the cohomological equations (3.35) and second, the de Rham cohomologies of degree  $D(n-1) - 1$  are trivial:

$$H^{D(n-1)-1}(\tilde{\mathcal{O}}_n) = H^{D(n-1)-1}(\mathcal{C}_{temp}^\infty(\mathbf{F}_n)) = \{0\}.$$

In fact, one possible strategy for solving Eqs. (3.35) on  $\tilde{\mathcal{O}}_n$  would be to find a homotopy operator

$$K_n : \Omega^{D(n-1)-1}(\tilde{\mathcal{O}}_n) \rightarrow \Omega^{D(n-1)-2}(\tilde{\mathcal{O}}_n), \quad K_n \circ d + d \circ K_n = \text{id},$$

and then a solution of (3.35) for  $n > 2$  will be

$$\mathbf{F}_n = \mathcal{F}_n \circ K_n,$$

where  $\mathcal{F}_n$  is the right hand side of Eq. (3.35) (as in (3.30)). Let us note that for the case of two dimensional QFT ( $D = 2$ ) the quadric  $x^2 = 0$



is reducible and the problem reduces to the case “ $D = 1$ ”, where such a differential extension  $\mathcal{O}_n \subseteq \tilde{\mathcal{O}}_n$  is provided by a *polylogarithmic extension* of the ring of rational functions  $\mathcal{O}_n$  (see e.g., [2]).

**Acknowledgments.** I am grateful to Spencer Bloch, Francis Brown, Maxim Kontsevich, Dirk Kreimer and Ivan Todorov for enlightening discussions. I am also grateful to Ivan Todorov for a critical reading of the manuscript. Parts of this work were developed during my stay at the Institut des Hautes Études Scientifiques in 2008 and also during the program “Operator algebras and conformal field theory” at the Erwin Schrödinger International Institute for Mathematical Physics. I am grateful to these institutions for the support and hospitality. This work was partially supported by the French–Bulgarian project Rila under the contract Egide – Rila N112.

## References

- [1] J.E. Björk, *Rings of differential operators*, North–Holland Publishing Company, Amsterdam at all 1979
- [2] F.C.S. Brown, Multiple zeta values and periods of moduli spaces  $\mathfrak{M}_{0,n}$ , math/0606419
- [3] R. Brunetti and K. Fredenhagen, Interacting quantum fields in curved space: renormalizability of  $\varphi^4$ , gr-qc/9701048
- [4] R. Brunetti and K. Fredenhagen, Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds, Commun. Math. Phys. **208** (2000), 623–661
- [5] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem, Comm. in Math. Phys., **210** (2000), 249–273, hep-th/9912092
- [6] A. Connes and D. Kreimer, Renormalization in quantum field theory and the Riemann–Hilbert problem. II. The beta-function, diffeomorphisms and the renormalization group, Comm. in Math. Phys. **216** (2001), 215–241, hep-th/0003188
- [7] H. Epstein and V. Glaser, The role of locality in perturbation theory, Ann. Henri Poincaré A XIX (1973), 211

- [8] E.R. Fadell and S.Y. Hussein, *Geometry and Topology of Configuration Spaces*, Springer
- [9] Jose M. Gracia-Bondia, Improved Epstein-Glaser renormalization in coordinate space I. Euclidean framework, hep-th/0202023
- [10] S. Hollands and R.M. Wald, Existence of Local Covariant Time Ordered Products of Quantum Fields in Curved Spacetime, Commun. Math. Phys. **231** (2002), 309-345
- [11] V.G. Kac, *Vertex algebras for beginners*, 2nd edition, University Lecture Series 10, Amer. Math. Soc., Providence, RI, 1998
- [12] D. Kreimer, On the Hopf algebra structure of perturbative quantum field theories, Adv. Theor. Math. Phys. **2** (1998), 303-334; q-alg/9707029
- [13] N.M. Nikolov, Vertex algebras in higher dimensions and globally conformal invariant quantum field theory, Commun. Math. Phys. **253** (2005), 283-322
- [14] N.M. Nikolov, Cohomological analysis of the Epstein-Glaser renormalization, arXiv:0712.2194
- [15] N.M. Nikolov and I.T. Todorov, Rationality of conformally invariant local correlation functions on compactified Minkowski space, Commun. Math. Phys. **218** (2001), 417-436
- [16] W. Rudin, *Functional analysis*, McGraw-Hill Book Company, New York at all 1973
- [17] O. Steinman, *Perturbation expansions in axiomatic field theory*, Lect. Notes in Physics **11**, Springer-Verlag, 1971
- [18] R. Stora, A note on elliptic perturbative renormalization on a compact manifold, unpublished notes, CERN & LAPP-TH.